B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

SÉRIE: RECHERCHES SUR LES DÉFORMATIONS

Volume LX, no. 2

ŁÓDŹ 2010

ŁÓDZKIE TOWARZYSTWO NAUKOWE
PL-90-505 Łódź, ul. M. Cuie-Skłodowskiej 11
tel. (42) 665 54 59, fax (42) 665 54 64
sprzedaż wydawnictw: tel. (42) 665 54 48
e-mail: ltn@ltn.lodz.pl

Wydano z pomocą finansową Ministerstwa Nauki i Szkolnictwa Wyższego

PL ISSN 0459–6854

Wydanie 1.
Nakład 250 egz.

Skład komputerowy: Zofia Fijarczyk

Druk i oprawa: *AnnGraf* s.c.
92-637 Łódź, ul. Rataja 54
tel. +48 601 24 10 83

BULLETIN DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ
2010 Vol. LX
Recherches sur les déformations no. 2, 126 pp.

FROM COMPLEX ANALYSIS TO ASYMMETRIC RANDERS-INGARDEN STRUCTURES IV

Editors of the Volume

Julian Ławrynowicz, Dariusz Partyka
and Józef Zając

Julian Ławrynowicz

Institute of Physics, University of Łódź
Pomorska 149/153, PL-90-236 Łódź, Poland
Institute of Mathematics, Polish Academy of Sciences
Łódź Branch, Banacha 22, PL-90-238 Łódź, Poland

Dariusz Partyka

Faculty of Mathematics and Natural Sciences
The John Paul II Catholic University of Lublin
Al. Racławickie 14, P.O. Box 129, PL-20-950 Lublin, Poland
State University of Applied Science in Chełm, Pocztowa 54
PL-22-100 Chełm, Poland, e-mail: partyka@kul.lublin.pl

Józef Zając

State University of Applied Science in Chełm
Pocztowa 54, PL-22-100 Chełm, Poland
Chair of Applied Mathematics
The John Paul II Catholic University of Lublin
Al. Racławickie 14, P.O. Box 129, PL-20-950 Lublin, Poland
e-mail: jzajac@kul.lublin.pl

Note. Since this year the numbering of volumes of the series *Recherches* coincides with that of the *Bulletin*. This permits to simplify the citations: Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **60**, no. 2 (2010) etc.

INSTRUCTION AUX AUTEURS

1. La présente Série du Bulletin de la Société des Sciences et des Lettres de Łódź comprend des communications du domaine des mathématiques, de la physique ainsi que de leurs applications liées aux déformations au sense large.
2. Toute communications est présentée à la séance d'une Commission de la Société par un des membres (avec deux opinions de spécialistes designés par la Rédaction). Elle doit lui être adressée directement par l'auteur.
3. L'article doit être écrit en anglais, français, allemand ou russe et débuté par un résumé en anglais ou en langue de la communication présentée. Dans tous les travaux écrits par des auteurs étrangers le titre et le résumé en polonais seront préparés par la rédaction. Il faut fournir le texte original qui ne peut contenir plus de 15 pages (plus 2 copies).
4. Comme des articles seront reproduits par un procédé photographique, les auteurs sont priés de les préparer avec soin. Le texte tapé sur un ordinateur de la classe IBM PC avec l'utilisation d'une imprimante de laser, est absolument indispensable. Il doit être tapé préférablement en *AMS-TEX* ou, exceptionnellement, en *Plain-TEX* ou *LATEX*. Après l'acceptation de texte les auteurs sont priés d'envoyer les disquettes (PC). Quelle que soient les dimensions des feuilles de papier utilisées, le texte ne doit pas dépasser un cadre de frappe de 12.3×18.7 cm (0.9 cm pour la page courante y compris). Les deux marges doivent être de la même largeur.
5. Le nom de l'auteur (avec de prénom complet), écrit en italique sera placé à la 1ère page, 5.6 cm au dessous du bord supérieur du cadre de frappe; le titre de l'acticle, en majuscules d'orateur 14 points, 7.1 cm au dessous de même bord.
6. Le texte doit être tapé avec les caractères Times 10 points typographiques et l'interligne de 14 points hors de formules longues. Les résumés, les rénvois, la bibliographie et l'adresse de l'auteurs doivent être tapés avec les petites caractères 8 points typographiques et l'interligne de 12 points. Ne laissez pas de "blancs" inutiles pour respecter la densité du texte. En commençant le texte ou une formule par l'alinéa il faut taper 6 mm ou 2 cm de la marge gauche, respectivement.
7. Les texte des théorèmes, propositions, lemmes et corollaires doivent être écrits en italique.
8. Les articles cités seront rangés dans l'ordre alphabétique et précédés de leurs numéros placés entre crochets. Après les références, l'auteur indiquera son adress complète.
9. Envoi par la poste: protégez le manuscript à l'aide de cartons.
10. Les auteurs recevront 20 tirés à part à titre gratuit.

Adresse de la Rédaction de la Série:
Département d'Analyse complexe et Géométrie différentielle
de l'Institut de Mathématiques de l'Académie polonaise des Sciences
BANACHA 22, PL-90-238 ŁÓDŹ, POLOGNE

Name and surname of the authors

TITLE – INSTRUCTION FOR AUTHORS SUBMITTING THE PAPERS FOR BULLETIN

Summary

Abstract should be written in clear and concise way, and should present all the main points of the paper. In particular, new results obtained, new approaches or methods applied, scientific significance of the paper and conclusions should be emphasized.

1. General information

The paper for BULLETIN DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ should be written in LaTeX, preferably in LaTeX 2e, using the style (the file **bull.cls**).

2. How to prepare a manuscript

To prepare the LaTeX 2e source file of your paper, copy the template file **instr.tex** with **Fig1.eps**, give the title of the paper, the authors with their affiliations/addresses, and go on with the body of the paper using all other means and commands of the standard class/style 'bull.cls'.

2.1. Example of a figure

Figures (including graphs and images) should be carefully prepared and submitted in electronic form (as separate files) in Encapsulated PostScript (EPS) format.

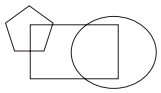


Fig. 1: The figure caption is located below the figure itself; it is automatically centered and should be typeset in small letters.

2.2. Example of a table

Tab. 1: The table caption is located above the table itself; it is automatically centered and should be typeset in small letters.

Description 1	Description 2	Description 3	Description 4
Row 1, Col 1	Row 1, Col 2	Row 1, Col 3	Row 1, Col 4
Row 2, Col 1	Row 2, Col 2	Row 2, Col 3	Row 2, Col 4

3. How to submit a manuscript

Manuscripts have to be submitted in electronic form, preferably via e-mail as attachment files sent to the address **zofija@mvii.uni.lodz.pl**. If a whole manuscript exceeds 2 MB composed of more than one file, all parts of the manuscript, i.e. the text (including equations, tables, acknowledgements and references) and figures, should be ZIP-compressed to one file prior to transfer. If authors are unable to send their manuscript electronically, it should be provided on a disk (DOS format floppy or CD-ROM), containing the text and all electronic figures, and may be sent by regular mail to the address: **Department of Solid State Physics, University of Lodz, Bulletin de la Société des Sciences et des Lettres de Łódź, Pomorska 149/153, 90-236 Lodz, Poland**.

References

[1]

Affiliation/Address

TABLE DES MATIÈRES

Left to right: Julian Ławrynowicz (Łódź) and Bogdan Bojarski (Warszawa)

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2010 Vol. LX

Recherches sur les déformations no. 2

pp. 11–14

Dedicated to Professor Julian Ławrynowicz
on the occasion of his 70-th birthday

Yuri Zelinskii

CONTINUOUS MAPPINGS BETWEEN DOMAINS OF MANIFOLDS

Summary

The main question considered in this paper relates to the desire to establish with what the minimum one can limit the number of preimages of an arbitrary point of the image, if the global degree of a given mapping of two domains is known *a priori*. In addition, we assume that this mapping realizes this minimum. Estimates in one direction, namely the lower possible value of this minimum, where obtained by the author in 1975 and 2005.

1.

We say that a mapping $f : X \to Y$ of topological spaces is finite-to-one, if the inverse image of an arbitrary point contains a finite or empty set of points. In what follows we assume that on the considered topological spaces is given the structure of manifolds and we have continuous mappings of these manifolds, or their subdomains. Also, we suppose that the topological degree of mapping $\deg f$ [1] is defined.

Theorem 1 [3]. *Let $f : \overline{D} \to \overline{D}_1$ be a continuous mapping (D and D_1 are open domains in the corresponding manifolds M^n and N^n) such that*

1. *$f(\partial D) \cap f(D) = \emptyset$, and*

2. *$H_c^{n-1}(\partial D; Z_2) \neq 0$ and the mapping $f^* : H_c^{n-1}(f\partial D; Z_2) \to H_c^{n-1}(\partial D; Z_2)$ induced by the restriction $f|_{\partial D}$ is an epimorphism. Then either $f|_D$ is a homeomorphism, or there is a point $y \in \mathrm{Int}D_1$ having at least three preimages in D. Yet if the mapping f is zero-dimensional, then in the latter case the set $A = \{y \mid f^{-1}y$ consists of at least three points$\}$ has dimension n.*

Theorem 1 provides in particular an answer to one of the problems posed by A. Kosinski [2] for the mapping of an n-dimensional Möbius strip.

2.

A mapping of a domain is called *interior mapping* if the image of every open set is open, and the preimage of an arbitrary point consists of isolated points.

Theorem 2 [4]. *Every proper mapping of a domain of an n-dimensional manifold onto a domain of another n-dimensional manifold of degree k either is an interior mapping or there exists a point in the image that possesses not less than $|k|+2$ original preimages. If the restriction f to the interior of the domain is the zero-dimensional mapping, than in the latter case mentioned above, the set of points of the image which do not possess less than $|k|+2$ original preimages contains a subset of complete dimensionality n.*

We say that a mapping is of class K_m, if the inverse image of each point contains no more than m points. In the case when we wish to fix the spaces under the mapping, we use the notation $K_m(X, Y)$.

Let $X = M^n$ be a closed n-dimensional manifold, as well $Y = B^n$ be the ball in an n-dimensional Euclidean space. In [4] it is found that for mappings of arbitrary closed two-dimensional manifolds in 2-disc class $K_2(M^2, B^2)$ is not empty.

Consider a projective space $\mathbb{R}P^n$ as the ball B^n as identified with the antipodal points of the boundary sphere S^{n-1}. Then the mapping h, which is inside the sphere is given as

$$h(x_1, x_2, x_3, \dots, x_n) = (x_1, |x_2|, |x_3|, \dots, |x_n|),$$

but on the sphere it identifies all points of the form $(\pm x_1, \pm x_2, \pm x_3, \dots, \pm x_n)$ and has multiplicity 2^{n-1}.

Corollary 1. *The classes $K_m(\mathbb{R}P^n, B^n)$ are nonempty for $m > 2^{n-1}$.*

3.

According to [6] in the mapping h there are points of a local homeomorphism. Suppose, without loss of generality that x_0 is such a point. Choose a neighbourhood $U(x_0)$ which is mapped under the mapping h homeomorphic to a circular neighbourhood of the point $h(x_0)$. Assume that the ball B^n is embedded in an n-dimensional sphere S^n. If now in the neighbourhood $U(x_0)$, we change the mapping h to φh, where φ is the inversion relative to the boundary sphere $\partial hU(x_0)$, and in the complement to $U(x_0)$ we leave the mapping h, then in general we obtain a map g of the projective space $\mathbb{R}P^n$ to the whole sphere S^n, which is of degree $\deg g = 1$. It is obvious that the multiplicity of such a mapping as compared with the mapping h will increase no more than by one. Thus, we have the following statement:

Theorem 3. *In the class of mappings of the projective space on the sphere S^n of the degree one there is a mapping belonging to the class $K_m(\mathbf{R}P^n, S^n)$ with $m = 2^{n-1} + 1$.*

The question of reducing the assessment of m for $n > 2$ remains an open problem; for $n = 2$ our estimate is precise, as follows from Theorems 2 and 3.

Unsolved problems

1. Does there exist a mapping of the n-dimensional projective space on an n-dimensional sphere such that every point of the image has no more then three preimage points for $n \geq 3$?
2. Does there exist a mapping of the n-dimensional projective space in an n-dimensional sphere such that every point of the image has no more then two preimage points for $n \geq 3$?
3. Do there exist for every proper mapping $f : \overline{D} \to \overline{D}_1$ (D, D_1 being domains of n-dimensional manifolds) a proper mapping g homotopic to f such that every point of the image $g(D)$ has no more then $|\deg f| + 2$ preimage points?

Let f be a continuous mapping defined on the boundary of the domain D into the domain D_1 of an n-dimensional manifold. We also assume that f belongs to $K_k(\partial D, D_1)$.

4. Find conditions for the existence of a continuous extension of f to the whole domain, so that $f|\mathrm{Int}D$ be an interior mapping.
5. Find conditions for the existence of a continuous extension of f to the whole domain, which belongs to $K_m(D, D_1)$, where m is a finite number.
6. Find conditions for the existence of a continuous extension of f to the whole domain, which belongs to $K_m(D, D_1)$, where $m < k + 2$.

References

[1] A. K. Bakhtin, G. P. Bakhtina, Yu. B. Zelinskii, *Topological-algebraic structures and geometric methods in complex analysis*, Proceedings of Institute of Mathematics of NAS of Ukraine **73** (2008), 308 (in Russian); MR2467439.

[2] A. Kosiński, *On a problem of Steinhaus*, Fund. Math. **46** (1958), 47–59; MR0132539 (24 #A2379).

[3] Yu. B. Zelinskii, *Certain problems of A. Kosiński*, Fund. Math. **46** (1958), 47–59; Ukrainian Math. J. **27**, no. 4 (1975), 510–516; MR0391107 (52 #11929).

[4] Yu. B. Zelinskii, *On multiplicity of continuous mappings of domains*, Ukrainian Math. J. **57**, no. 4 (2005), 666–670; MR2192932 (2006h:58011).

[5] Yu. B. Zelinskii, *On the mapping of domains on manifolds*, Transactions of Institute of Mathematics National Academy of Sciences of Ukraine **5**, no. 1 (2008), 149–152 (in Russian).

[6] Yu. Yu. Trokhimchuk, A. V. Bondar, *The local degree of a zero-dimensional mapping*, in: Metrical Questions of the Theory of Functions and Mappings, no. 1 (Proc. Colloq. on the Theory of Quasiconformal Mappings and its Generalizations, Donetsk, 1968) pp. 221–241 Izdat. "Naukova Dumka", Kiev 1969 (in Russian), MR0278272 (43 #4003).

Institute of Mathematics
National Academy of Sciences of Ukraine
3 Tereshchenkivs'ka Str.
UA-01 601 Kyiv 4
Ukraine
e-mail: zel@imath.kiev.ua

Presented by Leon Mikołajczyk at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 16, 2009

O ODWZOROWANIACH CIĄGŁYCH OBSZARÓW NA ROZMAITOŚCIACH

S t r e s z c z e n i e

Podstawowe zagadnienie rozważane w tej pracy dotyczy uzyskania minimum liczb przeciwobrazów dowolnego punktu obrazu przy odwzorowaniu dwóch obszarów na siebie, przy czym znamy *a priori* stopień odwzorowania. Dodatkowo zakładamy, że rozważane odwzorowanie realizuje to minimum. Oszacowania z jednej strony, a mianowicie najmniejszej z możliwych wartości omawianego minimum, były otrzymane przez autora w latach 1975 i 2005.

PL ISSN 0459-6854

BULLETIN

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2010 Vol. LX

Recherches sur les déformations no. 2

pp. 15–31

Arezki Touzaline

ON THE SOLVABILITY OF A QUASISTATIC CONTACT PROBLEM FOR ELASTIC MATERIALS

Summary

We consider a quasistatic contact problem between a linear elastic body and a foundation. The contact is modelled with normal compliance such that the penetration is restricted with unilateral constraint and the associated version of Coulomb's law of dry friction. Under a smallness assumption on the friction coefficient, we establish the existence of a weak solution to the problem. The proof is based on arguments of time-dicretization, compactness and lower semicontinuity.

1. Introduction

Contact problems involving deformable bodies are quite frequent in the industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve a complicated surface phenomena, and are modeled with highly nonlinear initial boundary value problems. Taking into account various frictional contact conditions associated with behavior laws becoming more and more complex leads to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. A first attempt to study frictional contact problems within the framework of variational inequalities was made in [6]. The mathematical, mechanical and numerical state of the art can be found in [13]. In [9] we find a detailed analysis of the contact problem in linear elasticity with the mathematical and numerical studies. In this paper we consider a quasistatic contact problem between a linear elastic body and an obstacle say a foundation. The contact is modelled with normal compliance similar to the one in [8] such that the penetration is restricted with unilateral constraint and the associated version of Coulomb's law of dry friction. Under this compliance condition the interpenetration of the body's surface into the foundation is allowed and me justified by considering

the interpenetration and deformation of surface asperities. On the other hand we want to point out the physical interest of the model studied here. Indeed, before the apparition of [8], it was well known that any restriction of the penetration was made in the compliance models. However according to [8], the method presented here considers a compliance model in which the compliance term doesn't represent necessarily an important perturbation of the original problem without contact. This will help us to study the models, where a strictly limited penetration is performed with the limit procedure to the Signorini contact problem. Here we recall that for linear elastic materials the quasistatic contact problem using a normal compliance law has been studied in [1] by considering incremental problems and in [10] by another method using a time-regularization. The quasistatic contact problem with local or nonlocal friction has been solved respectively in [11] and in [4] by using a time-discretization. In [2] the quasistatic contact problem with Coulomb friction was solved by an established shifting technique used to obtain increased regularity at the contact surface and by the aid of auxiliary problems involving regularized friction terms and a so-called normal compliance penalization technique. In viscoelasticity, the quasistatic contact problem with normal compliance and friction has been solved in [12]. In [7] the authors resolve the quasistatic contact problems in viscoelasticity and viscoplasticity. Carrying out the variational analysis, the authors systymatically use results on elliptic and evolutionary variational inequalities, convex analysis, nonlinear equations with monotone operators, and fixed points of operators. In [5] a quasistatic unilateral contact problem with friction and adhesion was studied and an existence result of a weak solution was established for a friction coefficient sufficiently small. In this paper we propose a variational formulation written in the form of two variational inequalities. By means of Euler's implicit scheme as in [4], the quasistatic contact problem leads us to solve a well-posed variational inequality at each time step. Finally under a smallness assumption on the coefficient of friction we prove by using lower semicontinuity and compactness arguments that the limit of the discrete solution is a solution to the continuous problem.

2. Variational formulation

Let $\Omega \subset \mathbb{R}^d; (d = 2, 3)$, be a domain, with a Lipschitz boundary Γ, initially occupied by a linear elastic body. Γ is partitioned into three measurable parts such that $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where Γ_1, Γ_2, Γ_3 are disjoint open sets and $meas\,(\Gamma_1) > 0$. The body is subjected to volume forces of density φ_1, prescribed zero displacements and tractions φ_2 on the part Γ_1 and Γ_2, respectively. On Γ_3 the body is in unilateral and frictional contact with finite penetration with a foundation.

Under these conditions, the classical formulation of the mechanical problem of frictional contact of the linear elastic body is the following.

Problem P_1. Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$ such that

$$\sigma = \mathcal{E}\varepsilon(u) \quad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$div\sigma + \varphi_1 = 0 \text{ in } \Omega \times (0,T), \tag{2.2}$$

$$u = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{2.3}$$

$$\sigma\nu = \varphi_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{2.4}$$

$$u_\nu \le g,\ \sigma_\nu + p(u_\nu) \le 0,\ (\sigma_\nu + p(u_\nu))(u_\nu - g) = 0 \ \text{ on } \Gamma_3 \times (0,T), \tag{2.5}$$

$$\begin{cases} |\sigma_\tau| \le \mu p(u_\nu) \\ |\sigma_\tau| < \mu p(u_\nu) \Longrightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| = \mu p(u_\nu) \Longrightarrow \exists \lambda \ge 0 \text{ s.t. } \sigma_\tau = -\lambda \dot{u}_\tau \end{cases} \qquad \text{on } \Gamma_3 \times (0,T), \tag{2.6}$$

$$u(0) = u_0 \ \text{ in } \Omega. \tag{2.7}$$

Here (2.1) is the elastic constitutive law in which σ denotes the stress tensor and $\mathcal{E}$ the fourth order tensor of elasticity coefficients, (2.2) represents the equilibrium equation, (2.3) and (2.4) are the displacement-tractions boundary conditions and, finally, the function u_0 denotes the initial displacement. We make some comments on the contact conditions (2.5) and (2.6) in which σ_ν denotes the normal stress, p is a prescribed nonnegative function, u_ν is the normal displacement, g is a positive constant which denotes the maximum value of the penetration, σ_τ represents the tangential traction and $\dot{u}_\tau$ represents the tangential velocity. Indeed, when $u_\nu < 0$ i.e. when there is separation between the body and the obstacle then the condition (2.5) combined with hypothese (2.13) shows that the reaction of the foundation vanishes (since $\sigma_\nu = 0$). When $0 \le u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$ which means that the reaction of the foundation is uniquely determined by the normal displacement. When $u_\nu = g$ then $-\sigma_\nu \ge p(g)$ and σ_ν is not uniquely determined. We note then when $g = 0$ and $p = 0$ then the condition (2.5) becomes the classical Signorini contact condition without a gap

$$u_\nu \le 0,\ \sigma_\nu \le 0,\ \sigma_\nu u_\nu = 0,$$

and when $g > 0$ and $p = 0$, condition (2.5) becomes the classical Signorini contact condition with a gap:

$$u_\nu \le g,\ \sigma_\nu \le 0,\ \sigma_\nu (u_\nu - g) = 0.$$

The last two conditions are used to model the unilateral conditions with a rigid foundation.

Conditions (2.6) represent a version of Coulomb's law of dry friction in which p is a prescribed nonnegative function, the so-called friction bound. The tangential shear cannot exeed the maximal frictional resistance $\mu p(u_\nu)$. Then, if the strict

inequality is satisfied, the surface adheres to the foundation and is in the so-called stick state, and when equality is satisfied there is relative sliding, the so-called slip state. Examples of normal compliance functions can be found in [1, 8, 12].
In the study of the mechanical problem P_1 we adopt the following notations and hypotheses:

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

where

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}$$

and S_d denotes the space of second order symmetric tensors in $\mathbb{R}^d$. In (2.6) and below, a dot above a variable represents its derivative with respect to time.

To proceed with the variational formulation, we need some function spaces:

$$H = \left(L^2(\Omega)\right)^d, \; Q = \left\{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},$$

$$H_1 = \left(H^1(\Omega)\right)^d, \; Q_1 = \{\tau \in Q : \operatorname{div}\tau \in H\}.$$

H, Q are Hilbert spaces equipped with the respective inner products:

$$\langle u, v\rangle_H = \int_\Omega u_i v_i dx, \;\; \langle \sigma, \tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\},$$

and the set of admissible displacements fields given by

$$K = \{v \in V : v_\nu \le g \text{ on } \Gamma_3\},$$

where $g \ge 0$. Since $meas\ \Gamma_1 > 0$, the following Korn's inequality holds [6],

$$\|\varepsilon(v)\|_Q \ge c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.8}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product given by

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$$

and let $\|.\|_V$ be the associated norm. It follows from (2.8) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent and $(V, \|.\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $d_\Omega > 0$ depending only on the domain Ω, $\Gamma_1$and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \le d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.9}$$

For every $v \in H_1$, we denote by v_ν and v_τ the normal and the tangential components of v on Γ given by

$$v_\nu = v.\nu, \qquad v_\tau = v - v_\nu \nu,$$

where ν is a unit outward normal vector to Γ. We also denote by σ_ν and σ_τ the normal and tangential component of a function $\sigma \in Q_1$ defined by $\sigma_\nu = \sigma\nu.\nu$, $\sigma_\tau = \sigma - \sigma_\nu\nu$, and we recall that when σ is a regular function, the following Green's formula holds:

$$\langle \sigma, \varepsilon(v) \rangle_Q + \langle div\sigma, v \rangle_H = \int_\Gamma \sigma\nu.v da \ \forall v \in H_1.$$

In the study of the mechanical problem P_1, we assume that $\mathcal{E} = (\mathcal{E}_{ijkh}) : \Omega \times S_d \to S_d$ is a bounded symmetric positive definite fourth order tensor, i.e.

$$\text{(2.10)} \qquad \begin{cases} \bullet\ \mathcal{E}_{ijkh} \in L^\infty(\Gamma_3), 1 \le i,j,k,h \le d. \\ \bullet\ \mathcal{E}\sigma.\tau = \sigma.\mathcal{E}\tau,\ \forall \sigma, \tau \in S_d,\ a.e.\ \text{in}\ \Omega. \\ \bullet\ \text{There exists } \alpha > 0 \text{ such that} \\ \mathcal{E}\tau.\tau \ge \alpha |\tau|^2 \ \ \forall \tau \in S_d,\ a.e.\ \text{in}\ \Omega. \end{cases}$$

We define the bilinear form $a(.,.)$ on $V \times V$ by

$$a(u,v) = \int_\Omega \mathcal{E}\varepsilon(u).\varepsilon(v)\, dx.$$

It follows from (2.10) that a is continuous and coercive, that is,

$$\text{(2.11)} \qquad \begin{cases} (a)\ \text{there exists } M > 0 \text{ such that} \\ |a(u,v)| \le M \|u\|_V \|v\|_V \quad \forall u, v \in V, \\ (b)\ \text{there exists } m > 0 \text{ such that} \\ a(v,v) \ge m \|v\|_V^2 \quad \forall v \in V. \end{cases}$$

For every real Banach space $(X,\ \|.\|_X)$ and $T > 0$ we use the notation $C([0,T];X)$ for the space of continuous functions from $[0,T]$ to X; recall that $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X .$$

For $p \in [1,\infty]$ we use the standard notation of $L^p(0,T;V)$. We also use the Sobolev space $W^{1,\infty}(0,T;V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T:V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)},$$

where a dot now represents the weak derivative with respect to the time variable. The forces are assumed to satisfy

$$\text{(2.12)} \qquad \varphi_1 \in W^{1,\infty}(0,T;H), \qquad \varphi_2 \in W^{1,\infty}\left(0,T;\left(L^2(\Gamma_2)\right)^d\right).$$

Let $f : [0,T] \to V$ given by

$$(f(t), v)_V = \int_\Omega \varphi_1 . v dx + \int_{\Gamma_2} \varphi_2 . v da \quad \forall\, v \in V,\ t \in [0,T].$$

The assumption (2.12) implies that

$$f \in W^{1,\infty}(0,T;V).$$

We assume that the contact function p satisfies

$$\begin{cases} (a)\ \ p :]-\infty, g] \to \mathbb{R}_+; \\ \\ (b)\ \text{there exists } L_p > 0 \text{ such that} \\ \qquad |p(u) - p(v)| \le L_p |u - v|, \ \text{ for all } u,\, v \le g; \\ \\ (c)\ \ (p(u) - p(v))(u - v) \ge 0, \text{ for all } u,\, v \le g; \\ \\ (d)\ \ p(v) = 0 \text{ for all } v \le 0. \end{cases} \tag{2.13}$$

Next, we define the functionals

$$j_\nu : V \times V \to \mathbb{R},\ \ j_\tau : V \times V \to \mathbb{R},$$

by

$$j_\nu(v,w) = \int_{\Gamma_3} p(v_\nu)\, w_\nu da,\ j_\tau(v,w) = \int_{\Gamma_3} \mu p(v_\nu) |w_\tau| da,$$

and let $j = j_\nu + j_\tau$. We suppose that the friction coefficient μ satisfies

$$\mu \in L^\infty(\Gamma_3) \ \text{ and } \mu \ge 0 \ \ a.e. \text{ on } \Gamma_3. \tag{2.14}$$

Also we assume that the initial data u_0 satisfies

$$u_0 \in K,\ a(u_0, v - u_0) + j(u_0, v - u_0) \ge (f(0), v - u_0)_V \quad \forall v \in K. \tag{2.15}$$

In the sequel, everywhere below c will denote a positive constant which does not depend on $n \in \mathbf{N}^*$ and $t \in [0,T]$ and whose value may change from line to line.

Now, in order to establish the weak formulation of Problem P_1, we assume that u is a smooth function satisfying (2.1) – (2.7). Indeed, let $v \in V$ and multiply the equilibrium of forces (2.2) by $v - \dot{u}(t)$, integrate the result over Ω and use Green's formula to obtain

$$\int_\Omega \sigma(t)(\varepsilon(v) - \varepsilon(\dot{u}(t)))\, dx = \int_\Omega \varphi_1(t).(v - \dot{u}(t)) dx + \int_\Gamma \sigma(t)\, \nu.(v - \dot{u}(t))\, da.$$

Taking into account of (2.4) and $v = 0$ on Γ_1, we see that

$$\int_\Gamma \sigma(t)\, \nu.(v - \dot{u}(t))\, da = \int_{\Gamma_2} \varphi_2(t).(v - \dot{u}(t)) da + \int_{\Gamma_3} \sigma(t)\, \nu.(v - \dot{u}(t))\, da.$$

Moreover we have

$$\int_{\Gamma_3} \sigma(t)\,\nu.\,(v-\dot{u}(t))\,da = \int_{\Gamma_3} \sigma_\nu(t)\,(v_\nu-\dot{u}_\nu(t))\,da + \int_{\Gamma_3} \sigma_\tau(t)\,(v_\tau-\dot{u}_\tau(t))\,da,$$

and

$$\int_{\Gamma_3} \sigma_\nu(t)\,(v_\nu-\dot{u}_\nu(t))\,da = \int_{\Gamma_3} (\sigma_\nu(t)+p(u_\nu(t)))\,(v_\nu-\dot{u}_\nu(t))\,da$$

$$-\int_{\Gamma_3} p(u_\nu(t))\,(v_\nu-\dot{u}_\nu(t))\,da.$$

The law of friction (2.6) leads to the following relation:

$$\sigma_\tau(v_\tau-\dot{u}_\tau)+\mu p(u_\nu)\,(|v_\tau|-|\dot{u}_\tau|)\geq 0 \quad \forall v_\tau,$$

from which we deduce that the function u satisfies the inequality

$$a(u(t),v-\dot{u}(t))+j(u(t),v)-j(u(t),\dot{u}(t))\geq (f(t),v-\dot{u}(t))_V$$

$$+\int_{\Gamma_3} (\sigma_\nu(t)+p(u_\nu(t)))\,(v_\nu-\dot{u}_\nu(t))\,da \quad \forall v\in V.$$

On the other hand we have

$$\int_{\Gamma_3} (\sigma_\nu(u(t))+p(u_\nu(t))\,(z_\nu-u_\nu(t))da =$$

$$\int_{\Gamma_3} (\sigma_\nu(u(t))+p(u_\nu(t)))((z_\nu-g)-(u_\nu(t)-g))da =$$

$$\int_{\Gamma_3} (\sigma_\nu(u(t))+p(u_\nu(t)))(z_\nu-g)da - \int_{\Gamma_3} (\sigma_\nu(u(t))+p(u_\nu(t))\,(u_\nu(t)-g)da.$$

Using the conditions (2.5) it follows that

$$\int_{\Gamma_3} (\sigma_\nu(u(t))+p(u_\nu(t))\,(z_\nu-g)da\geq 0 \quad \forall z\in K,$$

and

$$\int_{\Gamma_3} (\sigma_\nu(u(t))+p(u_\nu(t)))(u_\nu(t)-g)da = 0.$$

Hence from the preceding we deduce that

$$\int_{\Gamma_3} (\sigma_\nu(u(t))+p(u_\nu(t)))(z_\nu-u_\nu(t))\geq 0 \quad \forall z\in K.$$

Next, $\langle .,. \rangle$ shall denote the duality pairing on $H^{\frac{1}{2}}(\Gamma)$, $H^{-\frac{1}{2}}(\Gamma)$. We define the normal stress σ_ν as follows. Let $u \in H_1$ such that $div\sigma(u) = -\phi$ where $\phi \in H$. Then $\sigma_\nu(u) \in H^{-\frac{1}{2}}(\Gamma)$ is given by

$$\begin{cases} \langle \sigma_\nu(u), v_\nu \rangle = a(u,v) - (\phi, v)_H \\ \\ \forall v \in H_1 \text{ such that } v_\tau = 0 \text{ on } \Gamma. \end{cases} \tag{2.16}$$

We shall use the notation $\langle \theta\sigma_\nu, v_\nu \rangle = \langle \sigma_\nu, \theta v_\nu \rangle$, $\forall \theta \in C_0^1(\mathbb{R}^d)$. As in [5] let us introduce the function $\psi \in C_0^\infty(\mathbb{R}^d)$, $0 \leq \psi \leq 1$, such that $\psi = 1$ on $\bar{\Gamma}_3$ and 0 in a neighbourhoud of an open subset Γ_4 with $\text{supp}(\varphi_2(t)) \subset \Gamma_4 \subset \bar{\Gamma}_4 \subset \Gamma_2$ $\forall t \in [0,T]$.

Finally, with these notations we obtain a variational formulation of the problem P_1 as follows.

Problem P_2. Find a displacement field $u \in W^{1,\infty}(0,T;V)$ such that $u(0) = u_0$, $u(t) \in K$, for all $t \in [0,T]$, and for almost all $t \in (0,T)$,

$$\begin{aligned} a(u(t), v - \dot{u}(t)) + j(u(t), v) - j(u(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V \\ + \langle \sigma_\nu(u(t)) + p(u_\nu(t)), \psi(v_\nu - \dot{u}_\nu(t)) \rangle \quad \forall v \in V, \end{aligned} \tag{2.17}$$

and

$$\langle \sigma_\nu(u(t)) + p(u_\nu(t)), \psi(z_\nu - u_\nu(t)) \rangle \geq 0 \quad \forall z \in K. \tag{2.18}$$

One has the following

Theorem 2.1. *Let* (2.11), (2.12), (2.13), (2.14) *and* (2.15) *hold. Then, there exists a constant* $\mu_0 > 0$ *such that Problem* P_2 *has at least one solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0.$$

Remark 2.2. It is interesting to note that as in [4] any element u such that $u(t) \in K$ for all $t \in [0,T]$ and satisfying inequality (2.18) verifies

$$\left\langle \sigma_\nu(u(t)) + p(u_\nu(t)), \psi \frac{u_\nu(t+\Delta t) - u_\nu(t)}{\Delta t} \right\rangle \geq 0$$

and

$$\left\langle \sigma_\nu(u(t)) + p(u_\nu(t)), \psi \frac{u_\nu(t-\Delta t) - u_\nu(t)}{-\Delta t} \right\rangle \leq 0$$

for all $\Delta t > 0$. Moreover using the assumption (2.13) on p, one obtains that when $\Delta t \to 0$,

$$\langle \sigma_\nu(u(t)) + p(u_\nu(t)), \psi \dot{u}_\nu(t) \rangle = 0 \quad a.e.\ t \in (0,T). \tag{2.19}$$

3. Incremental formulation

For the proof of Theorem 2.1, we carry a time-discretization of Problem P_2. We need a partition of the time interval $[0,T]$, with $0 = t_0 < t_1 < ... < t_n = T$, where $t_i = i\Delta t$, $i = 0, ..., n$, with step size $\Delta t = T/n$. We denote by u^i the approximation

of u at time t_i and $\Delta u^i = u^{i+1} - u^i$. For a function $v \in C([0,T];X)$ where X is a Banah space we use the notation $v^i = v(t_i)$. By using an implicit scheme, we obtain a sequence of incremental problems P_n^i defined for $u^0 = u_0$ by

Problem P_n^i. Find $u^{i+1} \in K$ such that

(3.20)
$$\begin{cases} a\left(u^{i+1}, w - u^{i+1}\right) + j\left(u^{i+1}, w - u^i\right) - j\left(u^{i+1}, \Delta u^i\right) \\ \\ \geq \left(f^{i+1}, w - u^{i+1}\right)_V + \left\langle \sigma_\nu\left(u^{i+1}\right) + p\left(u_\nu^{i+1}\right), \psi(w_\nu - u_\nu^{i+1})\right\rangle \quad \forall\, w \in V, \\ \\ \left\langle \sigma_\nu\left(u^{i+1}\right) + p\left(u_\nu^{i+1}\right), \psi(w_\nu - u_\nu^{i+1})\right\rangle \geq 0 \quad \forall\, w \in K. \end{cases}$$

As in [5] Problem P_n^i is equivalent to Problem Q_n^i defined as follows.

Problem Q_n^i. Find $u^{i+1} \in K$ such that

(3.21)
$$\begin{cases} a\left(u^{i+1}, w - u^{i+1}\right) + j\left(u^{i+1}, w - u^i\right) - j\left(u^{i+1}, \Delta u^i\right) \\ \\ \qquad \geq \left(f^{i+1}, w - u^{i+1}\right)_V \quad \forall\, w \in K. \end{cases}$$

We have the following result.

Proposition 3.1. *There exists a constant $\mu_0 > 0$ such that Problem Q_n^i has a unique solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0.$$

To prove this proposition, we introduce the following intermediate problem.

Problem $Q_{n\eta}^i$. For $\eta \in K$, find $u_\eta^{i+1} \in K$ such that

(3.22)
$$\begin{cases} (Au_\eta^{i+1}, w - u_\eta^{i+1})_V + j_\tau\left(\eta, w - u^i\right) - j_\tau\left(\eta, u_\eta^{i+1} - u^i\right) \\ \\ \qquad \geq \left(f^{i+1}, w - u_\eta^{i+1}\right)_V \quad \forall\, w \in K, \end{cases}$$

where the operator $A : V \to V$ is defined as

$$(Au, v)_V = a(u, v) + j_\nu(u, v).$$

We can prove the following

Lemma 3.2. *Problem $Q_{n\eta}^i$ has a unique solution.*

Proof. We use (2.11) (a), (2.11) (b), (2.13) (b) and (2.13) (c) to see that the operator A is strongly monotone and Lipschitz continuous. The functional j_η defined on K by $j_\eta(w) = j_\tau\left(\eta, w - u^i\right)$ is proper convex and lower semicontinuous. From the theory of elliptic variational inequalities [3], it follows that the inequality (3.3) has a unique solution.

Now to prove Proposition 3.1, we define the following mapping

$$S : K \to K,$$

as

$$\eta \to S(\eta) = u_\eta.$$

The following lemma holds.

Lemma 3.3. *S has a unique fixed point η^* and u_{η^*} is a unique solution of Problem Q_n^i.*

Proof. We set $v = u_{\eta_s}$ in inequality of Problem $Q^i_{n\eta_r}$ and $v = u_{\eta_r}$ in inequality of Problem $Q^i_{n\eta_s}$. After adding the resulting inequalities, we obtain by using the hypthese $(2.13)\,(c)$ on p that

$$\begin{aligned} &a(u_{\eta_1} - u_{\eta_2}, u_{\eta_1} - u_{\eta_2}) \\ &\leq j_\tau\left(\eta_1, u_{\eta_2} - u^i\right) - j_\tau\left(\eta_1, u_{\eta_1} - u^i\right) + j_\tau\left(\eta_2, u_{\eta_1} - u^i\right) - j_\tau\left(\eta_2, u_{\eta_2} - u^i\right). \end{aligned}$$

By using $(2.11)\,(b)$, (2.9) and $(2.12)\,(b)$, we get

$$\|S(u_{\eta_2}) - S(u_{\eta_1})\|_V \leq \frac{d_\Omega^2}{m} L_p \|\mu\|_{L^\infty(\Gamma_3)} \|\eta_2 - \eta_1\|_V.$$

Let

$$\mu_0 = \frac{m}{d_\Omega^2 L_p}.$$

Then it follows that for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, S is a contraction; then it admits a unique fixed point η^* and u_{η^*} is a unique solution to Problem Q_n^i.

4. Existence result

The main result of this section is to show the existence of a solution obtained as a limit of the interpolate function of the discrete solution. For thus it is necessary at first to establish the following

Lemma 4.1. *For $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, we have*

$$\left\|u^{i+1}\right\|_V \leq c\left\|f^{i+1}\right\|_V, \quad \left\|\Delta u^i\right\|_V \leq c\left\|\Delta f^i\right\|_V. \tag{4.23}$$

Proof. Take $w = 0$ in inequality (3.2); then, using $(2.11)\,(b)$ and (2.9), we obtain by a standard reasoning the first inequality (4.1) for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$.

To prove the second inequality, set $v = u^i$ in inequality (3.2) and then $v = u^{i+1}$ in the translated inequality satisfied by u^i. We find after adding the resulting inequalities that

$$-a\left(\Delta u^i, \Delta u^i\right) + j\left(u^i, u^{i+1} - u^{i-1}\right) - j\left(u^i, u^i - u^{i-1}\right) - j\left(u^{i+1}, \Delta u^i\right)$$

$$\geq (-\Delta f^i, \Delta u^i)_V.$$

On the other hand we have

$$j\left(u^i, u^{i+1} - u^{i-1}\right) - j\left(u^i, u^i - u^{i-1}\right) - j\left(u^{i+1}, u^{i+1} - u^i\right) =$$

$$\int\limits_{\Gamma_3} \left(p\left(u_\nu^i\right) - p(u_\nu^{i+1})\right) \Delta u_\nu^i da + \int\limits_{\Gamma_3} \mu p\left(u_\nu^i\right) \left(\left|u_\tau^{i+1} - u_\tau^{i-1}\right| - \left|u_\tau^{i-1} - u_\tau^i\right|\right) da$$

$$- \int\limits_{\Gamma_3} \Delta u_\nu^i da - \int\limits_{\Gamma_3} \mu p\left(u_\nu^{i+1}\right) \left|\Delta u_\tau^i\right| da.$$

Moreover as

$$\int\limits_{\Gamma_3} \left(p\left(u_\nu^i\right) - p(u_\nu^{i+1})\right) \Delta u_\nu^i da \leq 0, \ \ \left|\left|u_\tau^{i+1} - u_\tau^{i-1}\right| - \left|u_\tau^{i-1} - u_\tau^i\right|\right| \leq \left|\Delta u_\tau^i\right|,$$

we obtain

$$a\left(\Delta u^i, \Delta u^i\right) \leq \int\limits_{\Gamma_3} \mu \left|p\left(u_\nu^{i+1}\right) - p\left(u_\nu^i\right)\right| \left|\Delta u_\tau^i\right| da + (\Delta f^i, \Delta u^i)_V.$$

Then using $(2.11)\,(b)$, $(2.13)\,(b)$ and (2.9), we get

$$m \left\|\Delta u^i\right\|_V^2 \leq d_\Omega^2 L_p \left\|\mu\right\|_{L^\infty(\Gamma_3)} \left\|\Delta u^i\right\|_V^2 + \left\|\Delta f^i\right\|_V \left\|\Delta u^i\right\|_V.$$

Therefore we obtain since

$$d_\Omega^2 L_p \left\|\mu\right\|_{L^\infty(\Gamma_3)} < m,$$

that

$$\left\|\Delta u^i\right\|_V \leq c \left\|\Delta f^i\right\|_V.$$

We shall now define the following sequence of functions:

$$u^n(t) = u^i + \frac{t - t_i}{\Delta t} \Delta u^i \ \ \forall t \in \ [t_i, t_{i+1}], \ i = 0, ..., n-1.$$

Then as in [7] we have the following

Lemma 4.2. *There exists a function u, such that passing to a subsequence still denoted (u^n) we have*

$$u^n \to u \ \textit{weak} \ * \ \textit{in} \ W^{1,\infty}(0, T; V).$$

On the other hand we introduce the following piecewise constant functions

$$\widetilde{u}^n : [0, T] \to V, \ \widetilde{f}^n : [0, T] \to V,$$

defined by

$$\widetilde{u}^n(t) = u^{i+1}, \ \widetilde{f}^n(t) = f(t_{i+1}), \ \forall\, t \in (t_i, t_{i+1}], \ i = 0, ..., n-1.$$

As in [4] the following result holds.

Lemma 4.3. *There exists a subsequence of* $(\widetilde{u}^n)$ *still denoted* $(\widetilde{u}^n)$ *such that the following result on convergence holds.*

$$\begin{aligned} &(i)\ \widetilde{u}^n \to u \text{ weak } * \text{ in } L^\infty(0,T;V), \\ &(ii)\ \widetilde{u}^n(t) \to u(t) \text{ weakly in } V \text{ a.e. } t \in [0,T], \\ &(iii)\ u(t) \in K \text{ for all } t \in [0,T]. \end{aligned} \tag{4.24}$$

Now we have all the ingredients to prove Theorem 2.1. To this end, we shall prove the following

Theorem 4.4. *The weak limit* u *of* $\widetilde{u}^n$ *satisfies the following inequality*: $\forall z \in L^2(0,T;V)$

$$\begin{cases} \displaystyle\int_0^T \left(a(u(t), z(t) - \dot{u}(t)) + j(u(t), z(t)) - j\left(u(t), \dot{u}(t)\right)\right) dt \\ \displaystyle\geq \int_0^T \left(f(t), z(t) - \dot{u}(t)\right)_V dt + \int_0^T \langle \sigma_\nu(u(t)) + p(u_\nu(t)), \psi z_\nu(t)\rangle\, dt \end{cases} \tag{4.25}$$

and satisfies the unilateral condition

$$\forall t \in [0,T] \quad \forall z \in K \quad \langle \sigma_\nu(u(t)) + p(u_\nu(t)), \psi(z_\nu - u_\nu(t))\rangle \geq 0. \tag{4.26}$$

Proof. In inequality (3.1) set, for $z \in V$, $w = u^i + z\Delta t$ and divide by Δt, we obtain

$$a\left(u^{i+1}, z - \frac{\Delta u^i}{\Delta t}\right) + j\left(u^{i+1}, z\right) - j\left(u^{i+1}, \frac{\Delta u^i}{\Delta t}\right) \geq \left(f^{i+1}, z - \frac{\Delta u^i}{\Delta t}\right)_V$$

$$+ \left\langle \sigma_\nu\left(u^{i+1}\right) + p\left(u_\nu^{i+1}\right), \psi\left(z_\nu - \frac{\Delta u_\nu^i}{\Delta t}\right)\right\rangle \quad \forall z \in V.$$

Since from the second inequality (3.1) we have

$$\left\langle \sigma_\nu\left(u^{i+1}\right) + p\left(u_\nu^{i+1}\right), \psi(z_\nu - \frac{\Delta u_\nu^i}{\Delta t})\right\rangle \geq \left\langle \sigma_\nu\left(u^{i+1}\right) + p\left(u_\nu^{i+1}\right), \psi z_\nu\right\rangle,$$

then it follows that

$$a\left(u^{i+1}, z - \frac{\Delta u^i}{\Delta t}\right) + j\left(u^{i+1}, z\right) - j\left(u^{i+1}, \frac{\Delta u^i}{\Delta t}\right) \geq$$

$$\left(f^{i+1}, z - \frac{\Delta u^i}{\Delta t}\right)_V + \left\langle \sigma_\nu\left(u^{i+1}\right) + p\left(u_\nu^{i+1}\right), \psi z_\nu\right\rangle \quad \forall z \in V.$$

This inequality implies that for any $z \in L^2(0,T;V)$:

$$\begin{cases} a(\widetilde{u}^n(t), z(t) - \dot{u}^n(t)) + j(\widetilde{u}^n(t), z(t)) - j(\widetilde{u}^n(t), \dot{u}^n(t)) \\ \geq \left(\widetilde{f}^n(t), z(t) - \dot{u}^n(t)\right)_V + \langle \sigma_\nu(\tilde{u}^n(t)) + p(\tilde{u}^n_\nu(t)), \psi z_\nu \rangle \text{ for } a.a.\ t \in (0,T). \end{cases}$$

Integrating both sides of the previous inequality on $(0,T)$ we obtain the inequality

(4.27)
$$\int_0^T a(\widetilde{u}^n(t), z(t) - \dot{u}^n(t))\,dt + \int_0^T j(\widetilde{u}^n(t), z(t))\,dt - \int_0^T j(\widetilde{u}^n(t), \dot{u}^n(t))\,dt$$
$$\geq \int_0^T \left(\widetilde{f}^n(t), z(t) - \dot{u}^n(t)\right)_V dt + \int_0^T \langle \sigma_\nu(\tilde{u}^n(t)) + p(\tilde{u}^n_\nu(t)), \psi z_\nu(t) \rangle\,dt.$$

Now before passing to the limit in the previous inequality we start with the proof of the following

Lemma 4.5. *The following properties hold:*

$$\liminf_{n\to\infty} \int_0^T a(\widetilde{u}^n(t), u^n(t))\,dt \geq \int_0^T a(u(t), \dot{u}(t))\,dt, \tag{4.28}$$

$$\lim_{n\to\infty} \int_0^T a(\widetilde{u}^n(t), z(t))\,dt = \int_0^T a(u(t), z(t))\,dt \quad \forall z \in L^2(0,T;V), \tag{4.29}$$

$$\lim_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), z(t))\,dt = \int_0^T j(u(t), z(t))\,dt \quad \forall z \in L^2(0,T;V), \tag{4.30}$$

$$\liminf_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), \dot{u}^n(t))\,dt \geq \int_0^T j\left(u(t), \dot{u}(t)\right)dt, \tag{4.31}$$

(4.32)
$$\lim_{n\to\infty} \int_0^T \left(\widetilde{f}^n(t), z(t) - \dot{u}^n(t)\right)_V dt = \int_0^T \left(f(t), z(t) - \dot{u}(t)\right)_V dt \quad \forall z \in L^2(0,T;V),$$

$$\lim_{n\to\infty}\int_0^T \langle \sigma_\nu(\tilde{u}^n(t)) + p(\tilde{u}_\nu^n(t)), \psi z_\nu(t)\rangle\, dt$$

$$= \int_0^T \langle \sigma_\nu(u(t)) + p(u_\nu(t)), \psi z_\nu(t)\rangle\, dt \quad \forall z \in L^2(0,T;V). \tag{4.33}$$

Proof. To prove (4.6) and (4.7) it suffices to see [4]. To show (4.8) we write that for $t \in [0,T]$

$$j(\widetilde{u}^n(t), z(t)) = (j(\widetilde{u}^n(t), z(t)) - j(u(t), z(t))) + j(u(t), z(t)).$$

Since

$$j(\widetilde{u}^n(t), z(t)) - j(u(t), z(t)) = \int_{\Gamma_3} (p(\tilde{u}_\nu^n(t)) - p(u_\nu(t)))\, z_\nu(t)\, da$$

$$+ \int_{\Gamma_3} \mu\,(p(\tilde{u}_\nu^n(t)) - p(u_\nu(t)))\, |z_\tau(t)|\, da,$$

by (2.14) (b) and (2.9), we get

$$\left| \int_0^T (j(\widetilde{u}^n(t), z(t)) - j(u(t), z(t)))\, dt \right| \le c\, \|\widetilde{u}_\nu^n - u_\nu\|_{L^2(0,T;L^2(\Gamma_3))}\, \|z\|_{L^2(0,T;V)},$$

so, since $\widetilde{u}_\nu^n \to u_\nu$ strongly in $L^2\left(0,T;L^2(\Gamma_3)\right)$, we get

$$\lim_{n\to\infty}\int_0^T (j(\widetilde{u}^n(t), z(t)) - j(u(t), z(t)))\, dt = 0,$$

and then (4.8) follows. Now to prove (4.9) we notice that

$$\int_0^T j(\widetilde{u}^n(t), \dot{u}^n(t))\, dt = \int_0^T (j(\widetilde{u}^n(t), \dot{u}^n(t)) - j(u(t), \dot{u}^n(t)))\, dt + \int_0^T j(u(t), \dot{u}^n(t))\, dt.$$

The previous equality implies

$$\left| \int_0^T (j(\widetilde{u}^n(t), \dot{u}^n(t)) - j(u(t), \dot{u}^n(t)))\, dt \right|$$

$$\le c\, \|\tilde{u}_\nu^n - u_\nu\|_{L^2(0,T;L^2(\Gamma_3))}\, \|\dot{u}^n\|_{L^2(0,T:V)}.$$

As $\dot{u}^n$ is bounded in $L^2(0,T;V)$, it follows that

$$\lim_{n\to\infty}\int_0^T \left(j\left(\widetilde{u}^n(t),\dot{u}^n(t)\right)-j\left(u(t),\dot{u}^n(t)\right)\right)dt=0.$$

For the convergence of the other term, we set

$$p\left(u_\nu(t)\right)=r(t).$$

Keeping in mind the assumption (2.13) on the function p, it follows that

$$r\in C\left([0,T];L^2(\Gamma_3)\right) \text{ and } r(t)\geq 0\ \forall\ t\in[0,T].$$

Moreover let the function φ_r be defined by

$$\varphi_r(z)=\int_{\Gamma_3} rz_\nu da+\int_{\Gamma_3}\mu r\left|z_\tau\right|da.$$

Then φ_r is lower semicontinuous and we have

$$\liminf_{n\to\infty}\int_0^T \varphi_r\left(\dot{u}^n(t)\right)dt\geq\int_0^T\varphi_r\left(\dot{u}(t)\right)dt,$$

and hence we deduce (4.9). To show (4.10), we use $(4.2)(i)$ and that $\widetilde{f}^n\to f$ strongly in $L^2(0,T;V)$. Finally to prove (4.11) it suffices to use (2.16), $(2.14)(b)$ and (4.2).

Now we use the inequality (3.2) to deduce that

$$a\left(u^{i+1},w-u^{i+1}\right)+j\left(u^{i+1},w-u^{i+1}\right)\geq\left(f^{i+1},w-u^{i+1}\right)_V\quad\forall\ w\in K.$$

Hence we obtain for all $t\in]0,T]$ the inequality

$$a\left(\tilde{u}^n(t),w-\tilde{u}^n(t)\right)+j\left(\tilde{u}^n(t),w-\tilde{u}^n(t)\right)\geq\left(\tilde{f}^n(t),w-\tilde{u}^n(t)\right)_V\quad\forall\ w\in K.$$

Passing to the limit as $n\to+\infty$ we obtain by using $(4.2)(ii)$ and (2.15) that for all $t\in[0,T]$:

$$a\left(u(t),w-u(t)\right)+j\left(u(t),w-u(t)\right)\geq\left(f(t),w-u(t)\right)_V\quad\forall\ w\in K.$$

Using Green's formula we obtain (4.4) and then (2.18).

We now use Lemma 4.5 and passe to the limit in (4.5) to obtain

$$\text{(4.34)}\quad\begin{cases}\displaystyle\int_0^T\left(a\left(u(t),z(t)-\dot{u}(t)\right)+j\left(u(t),z(t)\right)-j\left(u(t),\dot{u}(t)\right)\right)dt\\ \displaystyle\geq\int_0^T\left(f(t),z(t)-\dot{u}(t)\right)_V dt+\int_0^T\left\langle\sigma_\nu\left(u(t)\right)+p\left(u_\nu(t)\right),\psi z_\nu(t)\right\rangle dt.\end{cases}$$

Now keeping in mind (2.19), one arrives at the inequality (4.3).

If we set in (4.12) $z\in L^2(0,T;V)$ defined by

$$z(s)=\begin{cases}v & \text{for } s\in[t,t+\lambda],\\ \dot{u}(s) & \text{elsewhere,}\end{cases}$$

we get

$$\frac{1}{\lambda}\int_t^{t+\lambda}\left(a\left(u\left(s\right),v-\dot{u}\left(s\right)\right)+j\left(u\left(s\right),v\right)-j\left(u\left(s\right),\dot{u}\left(s\right)\right)\right)ds$$

$$\geq \frac{1}{\lambda}\int_t^{t+\lambda}\left(f\left(s\right),v-\dot{u}\left(s\right)\right)_V ds+\frac{1}{\lambda}\int_t^{t+\lambda}\left\langle\sigma_\nu\left(u\left(s\right)\right)+p\left(u_\nu\left(s\right)\right),\psi(v_\nu-\dot{u}_\nu\left(s\right))\right\rangle ds$$

and passing to the limit we obtain the inequality (2.17).

5. Conclusion

For the problem in question we have proved the existence of a weak solution under a smallness of the friction coefficient. As it is known, the question of uniqueness of the solution remains still open.

References

[1] L.-E. Andersson, *A quasistatic frictional problem with normal compliance*, Nonlinear Anal. Th. Appl. **16**, no. 4 (1991), 347–369.

[2] L.-E. Andersson, *Existence results for quasistatic contact problems with Coulomb friction*, Appl. Math. Optim. **42** (2000), 169–202.

[3] H. Brezis, *Equations et inéquations non linéaires dans les espaces vectoriels en dualité*, Annales Inst. Fourier **18** (1968), 115–175.

[4] M. Cocu, E. Pratt, and M. Raous, *Formulation and approximation of quasistatic frictional contact*, Int. J. Engng. Sc. **34**, no. 7 (1996), 783–798.

[5] M. Cocu and R. Rocca, *Existence results for unilateral quasistatic contact problems with friction and adhesion*, Mathematical Modelling and Numerical Analysis, **34**, no. 5 (2000), 981–1001.

[6] G. Duvaut and J. L. Lions, *Les inéquations en mécanique et en physique*, Dunod, Paris 1972.

[7] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*. Studies in advanced Mathematics 30, Americal Mathematical Society and International Press, 2002.

[8] J. Jarušek and M. Sofonea, *On the solvability of dynamic elastic-visco-plastic contact problems*, Zeitschrift für Angewandte Mathematik and Mechanik (ZAMM) **88**, no. 1 (2008), 3–22.

[9] N. Kikuchi and T. J. Oden, *Contact problems in elasticity: a study of variational inequalities and finite element methods SIAM*, Philadelphia 1988.

[10] A. Klarbring, A. Mikelic, and M. Shillor, *A global existence result for the quasistatic problem with normal compliance*, Internat. Ser. Numer. Math. 101, Birkhäuser Verlag, Basel 1991, pp. 85–111.

[11] R. Rocca, *Existence of a solution for a quasistatic problem of unilateral contact with local friction*, CR. Acad. Sci Paris. Ser. 1, **328** (1999), 1253–1258.

[12] M. Rochdi, M. Schillor, and M. Sofonea, *Quasistatic viscoelastic contact with normal compliance and friction*, J. Elasticity **51** (1998), 105–126.

[13] M. Sofonea, W. Han, and M. Shillor, *Analysis and Approximations of Contact Problems with Adhesion or Damage*, Pure and Applied Mathematics 276, Chapman and Hall / CRC Press, Boca Raton, Florida 2006.

Laboratoire de Systèmes Dynamiques
Faculté de Mathématiques, USTHB
BP 32 EL ALIA
Bab Ezzouar, 16111, Algérie
e-mail: ttouzaline@yahoo.fr

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 26, 2010

O ROZWIĄZALNOŚCI ZAGADNIENIA STYKU QUASI-STATYCZNEGO DLA CIAŁ SPRĘŻYSTYCH

S t r e s z c z e n i e

Rozważamy zagadnienie styku quasi-statycznego między liniowym ciałem sprężystym a podłożem. Styk jest modelowany z uginaniem normalnym, tak że przenikanie jest ograniczone z więzami jednostronnymi przy zachodzeniu odpowiedniego prawa Coulomba o tarciu na sucho. Przy założeniu małości współczynnika tarcia, dowodzimy istnienia słabego rozwiązania zagadnienia. Dowód jest oparty na argumancie dyskretyzacji czasu, zawartości i półciągłości z dołu.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2010 Vol. LX

Recherches sur les déformations no. 2

pp. 33–39

Dedicated to Professor Leszek Wojtczak
on the occasion of his 70-th birthday

Jerzy H. Rutkowski and Claude Surry

MELTING AND RELATED PHENOMENA IN THIN LEAD FILMS

Summary

The calculations of melting properties for thin lead films are presented. It was found that the film melting temperature, T_m, is less than the bulk melting temperature $T_m^b = 600.7\,\mathrm{K}$ and it decreases with the decrease of film thickness d. For film thickness $d > 26\,\text{Å}$ the film melting is proceeded by the surface melting. The surface melting appears at $T_{\mathrm{sm}} = 417.7\,\mathrm{K}$. The T_{sm} for $d > 30\,\text{Å}$ is constant and it achieves the value $T_{\mathrm{sm}} = 420\,\mathrm{K}$, the same as in the case of the bulk surface melting temperature. The quasi-liquid layer thickness increases with the temperature and the law of increasing is the same as observed for the bulk lead sample.

1. Introduction

Melting properties of lead films with thickness $d = 50\,\text{Å}$ and $d = 100\,\text{Å}$ were considered in the previous paper [1]. It was then found that:

i) The film melting temperature T_m is lower than the bulk melting temperature $T_m^b = 600.7\,\mathrm{K}$ and it depends on the film thickness. For $d = 50\,\text{Å}$ and $d = 100\,\text{Å}$ this temperatures are $T_m = 512\,\mathrm{K}$ and $T_m = 560.5\,\mathrm{K}$, respectively.

ii) The film surface melting temperature $T_{\mathrm{sm}} = 420\,\mathrm{K}$, and it is independent of the film thickness. Its value is the same as T_{sm} for the bulk material.

iii) The quasi-liquid layer which appears on the film surfaces at T_{sm} increases with the increasing of the film temperature. Its growth does not depend on the film thickness and the law of growth is similar to that of the bulk material.

The present contribution is a continuation and extension of the work [1]. The melting properties of lead films for thickness up to $150\,\text{Å}$ are calculated. The considerations made in the case of thin films ($d < 30\,\text{Å}$) showed however new features

of the melting process. The physical model and methodological approach are the same as in the paper [1]. The main results were presented on the Professor Leszek Wojtczak Colloquium [2].

2. Theoretical background

The film of thickness d is situated in the (x, y) – plane in which the melting properties are homogeneous. Across the film thickness, in the direction z perpendicular to the film surface, the order parameter exhibits its profile $m = m(z)$.

In order to describe the film melting properties the classical Landau-Ginzburg type functional is used in the following form [e.g. 3]:

$$F(m(z)) = f_s(m_s) + \int_0^d \left[f(m) + \frac{1}{2} I \left(\frac{dm}{dz} \right)^2 \right]. \tag{1}$$

The surface free energy term $f_s(m_s)$ in the formula (1) is given by

$$f_s(m_s) = f_s(m_0, m_d) = \frac{1}{2}\alpha_1 m_0^2 + \frac{1}{2}\alpha_2 m_d^2, \tag{2}$$

where for the sake of simplicity it is assumed that

$$\alpha_1 = \alpha_2 = \alpha_s \quad \text{and} \quad m_0 = m_d = m_s \tag{3}$$

with $\alpha_s = \text{const.}$, whereas m_0 and m_d are the order parameters at the surfaces.

The bulk Gibbs free energy $f(m)$ in the formula (1) has the form proposed by L. Wojtczak [1]:

$$f(m) = 6\alpha \left[\frac{1}{4} m^4 - \frac{1}{3}(1 + m^*) m^3 + \frac{1}{2} m^* m^2 \right] + \Lambda, \tag{4}$$

where

$$m^* = \frac{1}{2} - \frac{\Lambda}{\alpha}, \qquad \Lambda = L_m \left(1 - \frac{T}{T_m^b} \right) \tag{5}$$

with $\alpha = \text{const.}$, whereas T_m^b and L_m are the bulk melting temperature and the latent heat of bulk melting, respectively.

The Gibbs free energy for the bulk material given in the formula (4) has two minima at $m = 0$ and $m = 1$ which correspond to the liquid and crystalline phases, respectively. The maximum at $m = m^*$ makes a border between the phases. Therefore we have

$$0 < m \leq m^* \quad \text{for quasi-liquid and} \tag{6a}$$

$$m^* \leq m < 1 \quad \text{for quasi-solid.} \tag{6b}$$

The liquid bulk Gibbs free energy $f(m = 0)$ depends on temperature T. It decreases from a positive to negative values with increase of temperature and vanishes at $T = T_m^b$. The solid bulk Gibbs free energy $f(m = 1) = 0$ at any temperature. In

this way the function $f(m)$ describes properly the bulk melting transition [cf. 1]. It is worthwhile to notice that m^* increases with increasing temperature T.

We can see that in the present model the two phases are described by only one function (4) instead of two different functions usually applied; when each phase is described by its own function.

The second term under the integral in the formula (1), proportional to the material constant I, refers to the gradient contribution due to Landau-Ginzburg type of the thermodynamic potential.

In the case of Pb the constants appearing in formulas (1), (2), (4), and (5) have their values $\alpha_s = 0.5208\,\mathrm{J/m^2}$, $\alpha = 3.9365 \cdot 10^8\,\mathrm{J/m^3}$, $L_m = 2.47 \cdot 10^8\,\mathrm{J/m^3}$, $T_m^b = 600.7\,\mathrm{K}$ and $I = 1.5624 \cdot 10^{-10}\,\mathrm{J/m}$ ([4], [5]).

The properties of melting are obtained by minimization of the functional (1) with respect to the order parameter m. The procedure is done by the Finite Elements Method (cf. [1], [6] and [7]). The solutions lead to the order parameter profile $m(z)$ for its given film thickness d and temperature T. Two physical solutions are found for films thick enough, $d > 16\,\text{Å}$ and temperature $T > 0.203\,T_m$: i) $m(z) = 0$ for liquid and ii) $m(z) \neq 0$ for solid phase. Under the condition (3) the solutions are symmetric and have the feature $m(z) = m(d - z)$. The order parameter m has the lowest values on the film surfaces. Deeper in the sample the values of m are greater but never exceed one. With increasing temperature the order parameter m decreases. The decreasing of m is more remarkable close to the film surface than in the middle of the sample (see [1] as well as the Fig. 2, and Fig. 3 below).

The film melting temperature T_m is obtained by the solution of the equation

$$F\left(m(z=0), T_m\right) = F\left(m(z), T_m\right). \tag{7}$$

As it was mentioned above, when the temperature is growing up, the order parameter at surface m_s decreases. Simultaneously, the potential barrier value m^* increases. These two factors cause the equality of the m_s and m^* at some temperature $T = T_{\mathrm{sm}}$:

$$m_s(T_{\mathrm{sm}}) = m^*(T_{\mathrm{sm}}). \tag{8}$$

The solution of Eq. (8) gives the surface melting temperature T_{sm}, it means the temperature in which the quasi-liquid layer appears on the film surface.

At temperature $T > T_{\mathrm{sm}}$ the equality of $m(z)$ and m^* takes place inside the sample at a point $z = z^*$. In this case the distance from the surface to the intersection point can be interpreted as the quasi-liquid layer thickness z^* which for given temperature can be found from the following condition:

$$m(z^*) = m^*. \tag{9}$$

3. Results

The thickness dependence of the film melting T_m and film surface melting T_{sm} temperatures are presented in Fig. 1. We can see that the film melting temperature T_m

is less than the Pb bulk melting temperature $T_m^b = 600.7\,\mathrm{K}$ and it decreases with decreasing film thickness d. The melting is observed for foil thickness $d \geq 16\,\text{Å}$. It appears at $T_m = 279\,\mathrm{K}$. The surface melting occurs for $d > 26\,\text{Å}$ at the temperature $T_{\mathrm{sm}} = 417.7\,\mathrm{K}$. The T_{sm} for $d > 30\,\text{Å}$ does not depend on the film thickness and it has value $T_{\mathrm{sm}} = 420\,\mathrm{K}$, which is the same as the bulk surface melting temperature.

The order parameter profiles $m(z)$ for different film thickness d at their surface melting temperatures $T_{\mathrm{sm}} = 420\,\mathrm{K}$ and their melting temperatures T_m are shown in Fig. 2 and Fig. 3, respectively. The order parameter m of the liquid phase equals zero, independently of the temperature and the film thickness. It is not drawn in Fig. 2 and Fig. 3. We can see that for films thick enough the $m(z)$ exhibits *plateau* (with $m \approx 1$) in the middle of sample. For thinner films the *plateau* transforms into a maximum with $m < 1$.

The quasi-layer thickness growth is one of the important characteristics of the surface melting. Its temperature behaviour depends on a type of interaction. In the case of bulk Pb the short-range interaction forces lead to the logarithmic law of the growth with temperature. This law was confirmed for film thickness 50 Å and 100 Å in [1]. In Fig. 4a we can see z^* as a function of T while the relation z^* as a function of $\ln(T_m^b/(T_m^b - T))$ is presented in Fig. 4b. The data in Fig. 4 are collected for different film thickness. We can see that the law of layer growth does not depend on film thickness. The linear dependence of the quasi-liquid layer thickness in the inverse logarithmic temperature-difference scale (Fig. 4b) confirms the occurrence of short-range interaction forces in the Pb thin films.

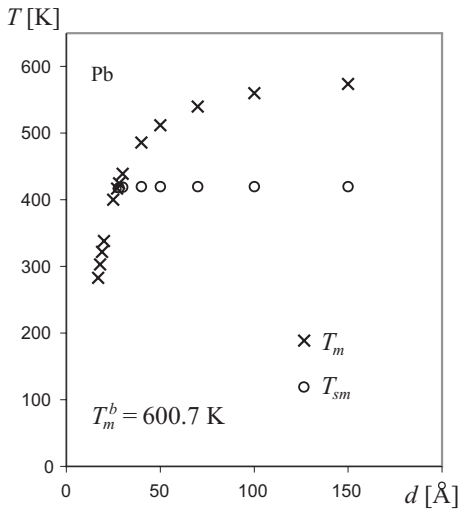


Fig. 1: The thickness dependence of the film melting T_m and surface melting T_{sm} temperatures.

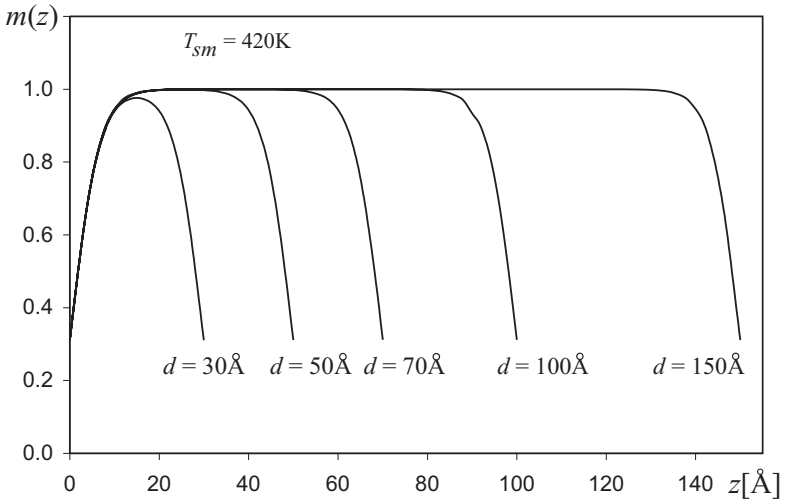


Fig. 2: The order parameter profiles $m(z)$ for different film thickness d at their surface melting temperatures $T_{\rm sm} = 420\,{\rm K}$.

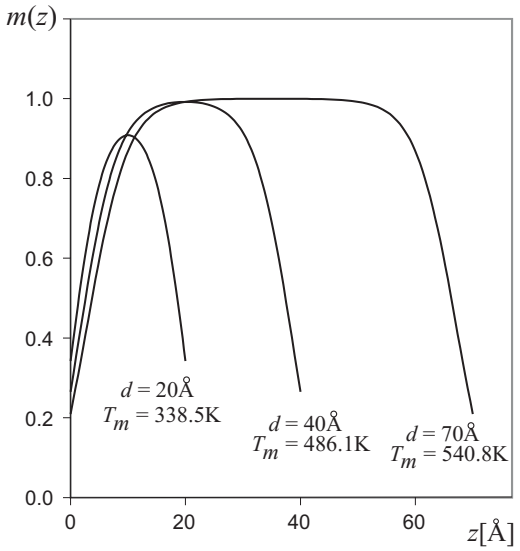


Fig. 3: The order parameter profiles $m(z)$ for different film thickness d at their melting temperatures T_m.

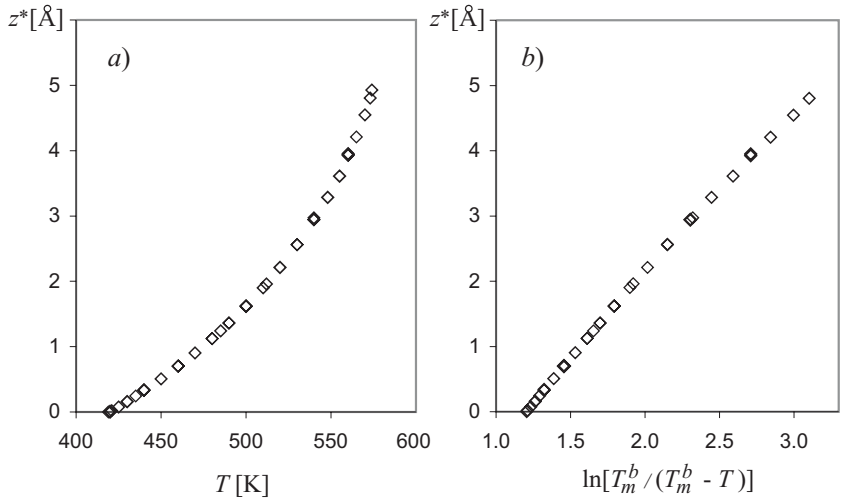


Fig. 4: The quasi-liquid layer thickness z^* as a function of temperature T, (Fig. 4a) and as a function of $\ln(T - m^b/(T_m^b - T))$, (Fig. 4b).

4. Conclusions

The calculations of the thin lead films melting properties confirmed the results obtained in the paper [1] and specified above in the Introduction. In addition it was found that the melting appears for films of thickness $d = 16\,\text{Å}$ at $T_m = 279\,\text{K}$. For the samples of thickness $d \leq 26\,\text{Å}$ the film melting is the first order phase transition whereas for $d > 26\,\text{Å}$ the film melting is proceeded by the surface melting – the second order phase transition. The surface melting appears at the temperature $T_{\text{sm}} = 417.7\,\text{K}$.

With increasing the film thickness the T_{sm} increases and for $d > 30\,\text{Å}$ it attains the value $T_{\text{sm}} = 420\,\text{K}$ typical for the bulk material. The quasi-liquid layer thickness z^* grows with increasing temperature T. For a chosen $T < T_m(d)$ the z^* does not depend on the film thickness d. The law of the quasi-liquid layer thickness growth with the film temperature confirms the existence of short-range interactions in the thin lead films.

The applied form of the $f(m)$ (the formula (4)) describes the melting process of Pb for the temperature greater than some critical one $T_c = 0.203T_m^b$. For $T < T_c$ the maximum of $f(m)$ appears at $m^* < 0$ which corresponds to the existence of the solid phase only. It is the reason that the solutions of Eq. (1) are limited to the film thickness $d < 16\,\text{Å}$. In spite of this, the mechanisms of the melting process presented above are still valid at least qualitatively.

References

[1] J. Rutkowski, L. Wojtczak, and C. Surry, Bull. Soc. Sci. Lettres Łódź **55** Ser. Rech. Deform. **46** (2005), 115.

[2] J. Rutkowski and C. Surry, Acta Physicae Superficierum **11** (2009), 114.

[3] R. Lipowsky, Phys. Rev. Letters **49** (1982), 1575.

[4] CRC Handbook of Chemistry and Physics, 75th Edition, ed. D. R. Lide, CRC Press, Boca Raton, 1994.

[5] B. Pluis, D. Frenkel, and J. F. van der Veen, Surf. Sci. **239** (1990), 282.

[6] C. Surry, J. Rutkowski, I. Zasada, and L. Wojtczak, Bull. Soc. Sci. Lettres Łódź **55** Ser. Rech. Deform. **46** (2005), 9.

[7] G. P. Nikishkov, Lecture Notes, UCLA 2001.

Department of Solid State Physics
University of Łódź
Pomorska 149/153, PL 90-236 Łódź
Poland

Laboratoire Félix Trombe
Institut de Sciences
et de Génie de Matériaux Procédés
CNRS, B. P. 5 Odeilo
F-66 125 Font Romeu Cédex
France

Presented by Claude Surry at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 26, 2010

TOPNIENIE I ZJAWISKA TOWARZYSZĄCE W CIENKICH PŁYTKACH OŁOWIU

S t r e s z c z e n i e

Przedstawione są wyliczenia własności topnienia cienkich płytek ołowiu. Pokazano, że temperatura topnienia płytki, T_m jest mniejsza od temperatury topnienia materiału masywnego $T_m^b = 600.7\,\mathrm{K}$ i rośnie wraz z grubością płytki d. Dla folii o grubościach $d > 26\,\text{Å}$ topnienie płytki jest poprzedzone topnieniem powierzchniowym. Zjawisko topnienia powierzchniowego pojawia się w temperaturze $T_{\mathrm{sm}} = 417.7\,\mathrm{K}$. Temperatura T_{sm} dla $d > 30\,\text{Å}$ jest stała i równa temperaturze topnienia powierzchniowego materiału masywnego, $T_{\mathrm{sm}} = 420\,\mathrm{K}$. Grubość warstwy kwasi-cieczy występującej na powierzchni płytki rośnie wraz ze wzrostem temperatury i wzrost ten jest taki sam jak w przypadku masywnej próbki ołowiu.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2010 Vol. LX

Recherches sur les déformations no. 2

pp. 41–46

Józef Zając and Beata Fałda

INFLUENCE OF PROFESSOR JULIAN ŁAWRYNOWICZ AND HIS ŁÓDŹ AND LUBLIN COLLEAGUES DURING 20 YEARS OF POLISH-MEXICAN COLLABORATION IN GENERALIZED COMPLEX ANALYSIS AND ITS APPLICATIONS

Summary

This presentation of common results obtained within Mexican-Polish Collaboration Agreement is going to express the reality and very truth character of this particular scientific activitity initiated by Professors Julian Ławrynowicz, José Adem, Enrique Ramírez de Arellano, Luis Manuel Tovar Sánchez, and Fray de Landa Castillo Alvarado.

1. Introduction

In summer 1989 two scientific agreements between Poland and Mexico were assigned in Poland. One of these agreements was established between Institute of Mathematics the Polish Academy of Sciences and Instituto Politecnico Nacional in Mexico City. The other one was assigned in Łódź between the University of Łódź and the Instituto Politecnico Nacional in Mexico City. These pair of scientific agreements appears to be very successful. Scientists participating in these two programs published more than 60 papers and developed real collaboration between different groups of Polish and Mexican researchers in physics, mathematics and biology. These facts show how effective and fruitfull has been this collaboration. The man who really initiated this activity was Professor Julian Ławrynowicz. Further on, in this collaboration we can see Professor Leszek Wojtczak, who assigned the second agreement as the Rector of the University of Łódź and paid a lot of care and attention to this activity, and Professors Jakub Rembieliński, Anna Urbaniak-Kucharczyk and Ilona Zasada. The 70th years anniversary of Professor Julian Ławrynowicz and 20 years of this

unusual cooperation is the best occasion to celebrate also a real friendship between Polish and Mexican people and institutions. Each year, two real leaders of this cooperation, namely Professor Julian Ławrynowicz and Professor Luis Manuel Tovar organized meetings, conferences and scientific workshops. In 1989 the first author of this presentation went to Mexico for the first time within this program and initiated very fruitful collaboration with Professor Lino Feliciano Reséndis Ocampo. Later the scientific programme was joined by other Lublin colleagues: Dariusz Partyla, Katarzyna Wilczek, Bogdan Paprocki, Agnieszka Wieczorek, and the second author of this presentation. Fascinating exotic country and very friendly Mexican people have been completing this collaboration. Most of our Mexican friends with their positive attitude to our scientific program makes us sure that this activity will have real future.

2. Scientific publications prepared by Julian Ławrynowicz and Mexican scientists

I. Variational problems for quasiconformal mappings

1. J. Ławrynowicz, E. Ramírez de Arellano, *Anti-involutions, symmetric complex manifolds, and quantum spaces*, in: Complex Analysis and Geometry. Ed. by V. Ancona, E. Ballico, and A. Silva (Lecture Notes in Pure and Applied Mathematics 173), Marcel Dekker Inc., New York-Basel-Hong Kong 1995, pp. 309–317.

II. Conformal and biholomorphic invariants

1. J. Ławrynowicz, L. F. Reséndis Ocampo, L. M. Tovar Sánchez, *Like-hyperbolic Bloch-Bergman classes*, Contemporary Mathematics (Anner. Math. Soc.) **509** (2010), 103–117.

III. Applications of variational problems in physics and optimal control

1. Ed. by J. Ławrynowicz, W. A. Rodrigues Jr., C. Surry, *Deformations of Mathematical Structures Applied in Physics*, in: Generalizations of Complex Analysis and Their Applications in Physics. Ed. by J. Ławrynowicz (Banach Center Publications 37), Institute of Mathematics, Polish Academy of Sciences, Warszawa 1996, pp. 241–356.

IV. Complex potential theory

1. J. Ławrynowicz, L. F. Reséndis Ocampo, L. M. Tovar Sánchez, *Q_p-hyperholomorphic functions and harmonic majorants in the quaternionic case*, Bull. Soc. Sci. Lettres Łódź **53** Sér. Rech. Déform. **41** (2003), 37–47.
2. J. Ławrynowicz, L. F. Reséndis Ocampo, L. M. Tovar Sánchez, *Q_p-hyperholomorphic functions and harmonic majorants in the hyper-kählerian case*, Rev. Roumaine Math. Pures Appl. **51** (2006), 689–702.

V. Applications of complex potential theory in physics

1. F. L. Castillo Alvarado, G. Contreras Puente, J. H. Rutkowski, L. Wojtczak, *A geometrical model of the pre-melting properties*, (a) VIII Latin-American Congress on Surface Sciences and Their Applications, Cancún 1994, Abstracts Booklet, SLACS, México, D.F., 1994, p. 100. Surfaces, Vacuum and Their Applications. Ed. by L. Hernández-Calderón and R. Asomoza, American Inst. of Phys. Press, Woodbury, NY, 1996, pp. 34–43.
2. F. L. Castillo Alvarado, J. Ławrynowicz, J. H. Rutkowski, L. Wojtczak, *Melting and surface melting. A survey*, Bull. Soc. Sci. Lettres Łódź **51** Sér. Rech. Déform. **35** (2001), 7–25.
3. F. L. Castillo Alvarado, A. Kazibudzka, J. Ławrynowicz, *Trade turnover and binary alloys as chaotic dynamical systems*, ibid. **52** Sér. Rech. Déform. **36** (2002), 67–84.
4. F. L. Castillo Alvarado, A. Sukiennicki, L. Wojtczak, I. Zasada, *Order-disorder phenomena in binary alloy thin films*, Physica B **344** (2004), 477–488.
5. L. Wojtczak, I. Zasada, A. Sukiennicki, F. L. Castillo Alvarado, *Surface melting and surface disordering in binary alloy thin films*, Phys. Rev. B **70** (2004), 195.416 (1–9).
6. A. Sukiennicki, L. Wojtczak, I. Zasada, F. L. Castillo Alavarado, *Ferromagnetic thin films of binary alloys*, JMMM **288** (2005), 137–145.
7. I. Zasada, A. Sukiennicki, L. Wojtczak, F. L. Castillo Alvarado, *Short-range order in AB_3 binary alloy thin films*, Phys. Rev. B **74** (2006), 2005402.
8. F. L. Castillo Alvarado, G. Contreras Puente, J. Ławrynowicz, J. H. Rutkowski, L. Wojtczak, *A physical background of geometrical model for the surface energy contribution*, Acta Physicae Superficierum **9** (2006), 129–148.

VI. Clifford analysis

1. J. Ławrynowicz, E. Ramírez de Arellano, J. Rembieliński, *The correspondence between type-reversing transformations of pseudo-euclidean Hurwitz pairs and Clifford algebras I*, (a) Dep. de Mat. Centro de Investigación y de Estudios Avanzados México Preprint no. 26 (1989), 53 pp., (b) Bull. Soc. Sci. Lettres Łódź **40**, no. 6 (Sér. Rech. Déform., Vol. "70th birthday's anniversary of Tadeusz Tietz" **8**, no. 76) (1990), 61–97.
2. J. Ławrynowicz, E. Ramírez de Arellano, J. Rembieliński, *The correspondence between type-reversing transformations of pseudo-euclidean Hurwitz pairs and Clifford algebras II*, (a) Dep. de Mat. Centro de Investigación y de Estudios Avanzados México Preprint no. 26 (1989), 53 pp, (b) Bull.

Soc. Sci. Lettres Łódź **40**, no. 7 (Sér. Rech. Déform., Vol. "70th birthday's anniversary of Tadeusz Tietz" **8**, no. 77) (1990), 88–129.

3. J. Ławrynowicz, R. M. Porter, E. Ramírez de Arellano, J. Rembieliński, *On dualities generated by the generalized Hurwitz problem*, (a) Dep. de Mat. Centro de Investigación y de Estudios Avanzados México Preprint no. 34 (1989), 23 pp., (b) Deformations of Mathematical Structures II. Hurwitz-type Structures and Applications to Surface Physics. Selected papers from the Seminar on Deformations, Łódź – Malinka 1988/92. Ed. by J. Ławrynowicz, Kluwer Academic Publishers, Dordrecht-Boston-London 1994, pp. 189–208.
4. J. Adem, J. Ławrynowicz, J. Rembieliński, *Generalized Hurwitz maps of the type* $S \times V \to W$, (a) Dep. de Mat. Centro de Investigación y de Estudios Avanzados Preprint no. 80 (1990), ii + 20 pp., (b) Rep. Math. Phys. **37** (1996), 325–336.
5. (a) J. Ławrynowicz, *Type-changing transformations of Hurwitz pairs, quasiregular functions, and hyperkählerian holomorphic chains II*, 7th Internat. Workshop on Complex Structures and Vector Fields Abstracts, Plovdiv 2004, p. 50, (b) J. Ławrynowicz, M. Nowak-Kępczyk, L. M. Tovar Sánchez, Ditto, in: Contemporary Aspects of Complex Analysis, Differential Geometry and Mathematical Physics. Ed. by S. Dimiev and K. Sekigawa, World Scientific, New Jersey-London-Singapore 2005, pp. 148–173.

VII. Applications of Clifford analysis in physics

1. J. Avendaño López, F. L. Castillo Alvarado, J. Ławrynowicz, R. A. Barrio, A. Urbaniak-Kucharczyk, *Clifford analysis, Riemannian geometry, and electromagnetic field*, (a) Dep. de Mat. Centro de Investigación y de Estudios Avanzados México Preprint no. 81 (1991), 19 pp., (b) Essays on the Formal Aspects of Electromagnetic Theory. Ed. by A. Lakhtakia, World Scientific, Singapore-New Jersey-London-Hong Kong 1993, pp. 533–558.
2. F. L. Castillo Alvarado, G. Contreras Puente, J. Ławrynowicz, L. Wojtczak, *A Hurwitz-pair approach to the pre-melting problem*, in: Deformations of Mathematical Structures II. Hurwitz-Type Structures and Applications to Surface Physics. Selected papers from the Seminar on Deformations, Łódź-Malinka 1988/92. Ed. by J. Ławrynowicz, Kluwer Academic Publishers, Dordrecht-Boston-London 1994, pp. 289–298.
3. J. Ławrynowicz, W. A. Rodrigues Jr., L. Wojtczak, *Clifford-algebraical approach to the particle lifetimes*, Bull. Soc. Sci. Lettres Łódź 46 Sér. Rech. Déform. **21** (1996), 57–63.
4. J. Ławrynowicz, *Clifford-analytical approach to stochastical geometry*, Internat. Conf. on Clifford Algebras and Their Applications in Mathe-

matical Physics, Ixtapa-Zihuatanejo, México 1999, Scientific Program, CIMAT, Guanajuato, Gto. 1999, p. 25.

5. F. L. Castillo Alvarado, G. Contreras Puente, J. Ławrynowicz, J. H. Rutkowski, L. Wojtczak, *The Clifford-algebraic structure for the surface melting phenomena*, IX Latin American Congress on Surface Science and Its Applications, La Habana 1999, Abstracts Booklet, La Habana 1999, p. 27.

6. J. Ławrynowicz, W. A. Rodrigues Jr., L. Wojtczak, *Stochastical electrodynamics in Clifford-analytical formulation related to entropy-depending structures*, Bull. Soc. Sci. Lettres Łódź **53** Sér. Rech. Déform. **41** (2003), 69–87.

7. F. L. Castillo Alvarado, G. Contreras Puente, J. Ławrynowicz, J. H. Rutkowski, L. Wojtczak, *The Clifford-algebraic structure for the surface melting phenomena*, Acta Physicae Superficierum **9** (2006), 109–128.

VIII. Fractal analysis

1. J. Ławrynowicz, O. Suzuki, F. L. Castillo Alvarado, *Basic properties and applications of graded fractal bundles related to Clifford structures: An introduction*, Ukrain. Math. Zh. **60** (2008), 603–618.

IX. Applications of fractal analysis in physics

1. F. L. Castillo Alvarado, J. Ławrynowicz, O. Suzuki, *Duality for fractal sets and lattice models on fractal sets* [*A noncommutative diferential-geometric method to fractal geometry (I)*], Bull. Soc. Sci. Lettres Łódź **53** Sér. Rech. Déform. **41** (2003), 49–68.

2. J. Ławrynowicz, O. Suzuki, F. L. Castillo Alvarado, *Basic properties and applications of graded fractal bundles related to Clifford structures: Scope of applications and inoculation vs. periodicity*, ibid. **57** Sér. Rech. Déform. **53** (2007), 37–52.

3. J. Ławrynowicz, O. Suzuki, F. L. Castillo Alvarado, *Basic properties and applications of graded fractal bundles related to Clifford structures: Physical applications*, ibid. **58** Sér. Rech. Déform. **55** (2008), 81–97.

4. F. L. Castillo Alvarado, J. Ławrynowicz, M. Nowak-Kępczyk, *Fractal modelling of alloy thin films. Temperatures characteristic for local phase transitions*, in: Applied Complex and Quaternionic Approximation. Ed. by Ralitza K. Kovacheva, J. Ławrynowicz, and S. Marchiafava, Ediz. Univ. "La Sapienza", Roma 2009, pp. 207–236.

5. J. Ławrynowicz, F.L. Castillo Alvarado, A. Niemczynowicz, *Some algebraical and geometrical aspects of simple orthorhombic Ising-Onsager-Zhang lattices*, Philosophical Magazine, 32 pp., to appear.

3. Scientific publications prepared by Józef Zając and Lino Reséndis Ocampo

1. L. Reséndis, J. Zając, *Area and linear distortion theorems for quasiconformal mappings*, Bull. Soc. Sci. Lettres, Łódź, **45** Sér. Rech. Déform. **20** (1995), 57–65.
2. L. Reséndis, J. Zając, *Maximal dilatation of the Beurling-Ahlfors extension operator*, Folia Sci. Univ. Tech. Resoviensis Math. **20** (154) (1996), 141–154.
3. J. Zając, *Special functions of Teichmüller theory*, Proc. of The International Symposium on Complex Analysis and Related Topics, Cuernevaca, México, D.F., 1997, p. 80.
4. L. Reséndis, J. Zając, *The Beurling-Ahlfors extension of quasihomographies*, Commentarii Mathematici Universitatis Sancti Pauli **46**, no. 2 (1997), 133–174.
5. L. Reséndis, W. Rompała, J. Zając, *Teichmüller structure of endomorphisms and dynamical systems*, Bull. Soc. Sci. Lettres, Łódź, **49** Sér. Rech. Déform. **28** (1999), 115–127.
6. L. Reséndis, J. Zając, *The Paprocki spaces*, Proc. of the 5th Environmental Mathematical Conference, vol. 1, The Catholic University Press, Lublin 1999, pp. 127–138.

State University of Applied Science in Chełm
Pocztowa 54, PL-22-100 Chełm
Chair of Applied Mathematics
The John Paul II Catholic University of Lublin
Al. Racławickie 14, P.O. Box 129, PL-20-950 Lublin
Poland
e-mail: jzajac@kul.lublin.pl
e-mail: bfalda@kul.lublin.pl

Presented by Leon Mikołajczyk at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 26, 2010

WPŁYW PROFESORA JULIANA ŁAWRYNOWICZA I JEGO ŁÓDZKICH I LUBELSKICH KOLEGÓW NA 20–LETNIĄ WSPÓŁPRACĘ POLSKO-MEKSYKAŃSKĄ W ZAKRESIE UOGÓLNIONEJ ANALIZY ZESPOLONEJ

S t r e s z c z e n i e

Obecna prezentacja wspólnych wyników uzyskanych w zakresie Polsko-Meksykańskiej Umowy o Współpracy wykazuje rzeczywisty i autentyczny charakter tej szczególnej aktywności naukowej zainicjowanej przez Profesorów Juliana Ławrynowicza, José Adema, Enrique Ramíreza de Arellano, Luisa Manuela Tovara Sáncheza i Fraya de Landa Castillo Alvarado.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2010 Vol. LX

Recherches sur les déformations no. 2

pp. 47–54

Vitaly S. Shpakivskyi and Sergeĭ A. Plaksa

INTEGRAL THEOREMS AND A CAUCHY FORMULA IN A COMMUTATIVE THREE-DIMENSIONAL HARMONIC ALGEBRA

Summary

For monogenic functions taking values in a three-dimensional commutative harmonic algebra with the unit and two-dimensional radical, we have proved analogs of classical integral theorems of the theory of analytic functions of the complex variable: the Cauchy integral theorems for surface integral and curvilinear integral, the Morera theorem and the Cauchy integral formula.

1. Introduction

Let $\mathbb{A}_3$ be a three-dimensional commutative associative Banach algebra with the unit 1 over the field of complex numbers $\mathbb{C}$. Let $\{1, \rho_1, \rho_2\}$ be a basis of the algebra $\mathbb{A}_3$ with the multiplication table $\rho_1\rho_2 = \rho_2^2 = 0$, $\rho_1^2 = \rho_2$.

The algebra $\mathbb{A}_3$ is *harmonic* (see [1,2]) because there exist *harmonic* bases $\{e_1 = 1, e_2, e_3\}$ in $\mathbb{A}_3$ satisfying the following condition

$$e_1^2 + e_2^2 + e_3^2 = 0. \tag{1}$$

Consider the linear envelope $E_3 := \{\zeta = x + ye_2 + ze_3 : x, y, z \in \mathbb{R}\}$ generated by the vectors $1, e_2, e_3$ over the field of real numbers $\mathbb{R}$. For a set $S \subset \mathbb{R}^3$ consider the set $S_\zeta := \{\zeta = x + ye_2 + ze_3 : (x, y, z) \in S\} \subset E_3$ congruent to S. In what follows, $\zeta = x + ye_2 + ze_3$ and $x, y, z \in \mathbb{R}$.

A continuous function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is *monogenic* in a domain $\Omega_\zeta \subset E_3$ if Φ is differentiable in the sense of Gateaux in every point of Ω_ζ, i. e. if for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbb{A}_3$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall h \in E_3.$$

It follows from the equality (1) and the equality

$$\triangle_3\Phi := \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = \Phi''(\zeta)(e_1^2 + e_2^2 + e_3^2)$$

that every twice monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ satisfies the three-dimensional Laplace equation $\Delta_3\Phi = 0$.

In the paper [3] for functions differentiable in the sense of Lorch in an arbitrary convex domain of commutative associative Banach algebra, some properties similar to properties of holomorphic functions of complex variable (in particular, the integral Cauchy theorem and the integral Cauchy formula, the Taylor expansion and the Morera theorem) are established. In the paper [4] the convexity of the domain is withdrawn in the mentioned results from [3].

In this paper we establish similar results for monogenic functions $\Phi : \Omega_\zeta \to \mathbb{A}_3$ given only in a domain Ω_ζ of the linear envelope E_3 instead of domain of the whole algebra $\mathbb{A}_3$. Let us note that *a priori* the differentiability of the function Φ in the sense of Gâteaux is a restriction weaker than the differentiability of this function in the sense of Lorch. Moreover, note that the integral Cauchy formula established in the papers [3, 4] is not applicable for a monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ because it deals with an integration along a curve on which the function Φ is not given, generally speaking.

Note that as well as in [3,4], some hypercomplex analogues of the integral Cauchy theorem for a curvilinear integral are established in the papers [5, 6]. In the papers [5, 7–9] similar theorems are established for surface integral.

2. Cauchy integral theorem for a surface integral

A function $\Phi(\zeta)$ of the variable $\zeta \in \Omega_\zeta$ is monogenic if and only if the following Cauchy-Riemann conditions are satisfied (see Theorem 1.3 [2]):

$$\frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial x}\, e_2, \qquad \frac{\partial\Phi}{\partial z} = \frac{\partial\Phi}{\partial x}\, e_3. \tag{2}$$

Along with monogenic functions, consider a function $\Psi : \Omega_\zeta \to \mathbb{A}_3$ having continuous partial derivatives of the first order in a domain Ω_ζ and satisfying the equation

$$\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y}\, e_2 + \frac{\partial\Psi}{\partial z}\, e_3 = 0 \tag{3}$$

at every point of this domain.

In the scientific literature the different denominations are used for functions satisfying equations of the form (3). For example, in [5, 6, 10] – regular functions, and in the papers [7, 8, 11] they are called monogenic functions. As well as in the papers [9, 12, 13], we call a function *hyperholomorphic* if it satisfies the equation (3).

It is well known that in the quaternionic analysis the classes of functions determined by means conditions of the form (2) and (3) do not coincide (see [5, 14]).

Note that in the algebra $\mathbb{A}_3$ the set of monogenic functions is a subset of the set of hyperholomorphic functions because every monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ satisfies the equality (3) owing to the conditions (1), (2). Yet, there exist hyperholomorphic functions which are not monogenic. For example, the function

$$\Psi(x + ye_2 + ze_3) = ze_2 - ye_3$$

satisfies the condition (3), but it does not satisfy the equalities of the form (2).

Let Ω be a bounded closed set in $\mathbb{R}^3$. For a continuous function $\Psi : \Omega_\zeta \to \mathbb{A}_3$ of the form

$$\Psi(x + ye_2 + ze_3) = \sum_{k=1}^{3} U_k(x, y, z)e_k + i\sum_{k=1}^{3} V_k(x, y, z)e_k, \tag{4}$$

where $(x, y, z) \in \Omega$, we define a volume integral by the equality

$$\int\limits_{\Omega_\zeta} \Psi(\zeta)dxdydz := \sum_{k=1}^{3} e_k \int\limits_{\Omega} U_k(x, y, z)dxdydz + i\sum_{k=1}^{3} e_k \int\limits_{\Omega} V_k(x, y, z)dxdydz.$$

Let Σ be a quadrable surface in $\mathbb{R}^3$ with quadrable projections on the coordinate planes. For a continuous function $\Psi : \Sigma_\zeta \to \mathbb{A}_3$ of the form (4), where $(x, y, z) \in \Sigma$, we define a surface integral on Σ_ζ with the differential form

$$\sigma_{\alpha_1,\alpha_2,\alpha_3} := \alpha_1 dydz + \alpha_2 dzdxe_2 + \alpha_3 dxdye_3, \text{ where } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R},$$

by the equality

$$\begin{aligned}\int\limits_{\Sigma_\zeta} \Psi(\zeta)\sigma_{\alpha_1,\alpha_2,\alpha_3} := \sum_{k=1}^{3} e_k \int\limits_{\Sigma} \alpha_1 U_k(x, y, z)dydz + \sum_{k=1}^{3} e_2e_k \int\limits_{\Sigma} \alpha_2 U_k(x, y, z)dzdx \\ + \sum_{k=1}^{3} e_3e_k \int\limits_{\Sigma} \alpha_3 U_k(x, y, z)dxdy + i\sum_{k=1}^{3} e_k \int\limits_{\Sigma} \alpha_1 V_k(x, y, z)dydz \\ + i\sum_{k=1}^{3} e_2e_k \int\limits_{\Sigma} \alpha_2 V_k(x, y, z)dzdx + i\sum_{k=1}^{3} e_3e_k \int\limits_{\Sigma} \alpha_3 V_k(x, y, z)dxdy.\end{aligned}$$

A connected homeomorphic image of a square in $\mathbb{R}^3$ is called *simple surface*. A surface is *locally-simple* if it is simple in a certain neighbourhood of every point.

If a simply connected domain $\Omega \subset \mathbb{R}^3$ has a closed locally-simple piecewise-smooth boundary $\partial\Omega$ and a function $\Psi : \Omega_\zeta \to \mathbb{A}_3$ is continuous together with partial derivatives of the first order up to the boundary $\partial\Omega_\zeta$, then the following analogue of the Gauss-Ostrogradski formula is true:

$$\int\limits_{\partial\Omega_\zeta} \Psi(\zeta)\sigma = \int\limits_{\Omega_\zeta} \left(\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y}e_2 + \frac{\partial\Psi}{\partial z}e_3\right) dxdydz, \tag{5}$$

where $\sigma := \sigma_{1,1,1} \equiv dydz + dzdxe_2 + dxdye_3$. Now, the next theorem is a result of the formula (5) and the equality (3).

Theorem 1. *Suppose that Ω is a simply connected domain with a closed locally-simple piecewise-smooth boundary $\partial\Omega$. Suppose also that the function $\Psi : \overline{\Omega_\zeta} \to \mathbb{A}_3$ is continuous in the closure $\overline{\Omega_\zeta}$ of the domain Ω_ζ and hyperholomorphic in Ω_ζ. Then*

$$\int\limits_{\partial\Omega_\zeta} \Psi(\zeta)\sigma = 0.$$

3. Cauchy integral theorem for a curvilinear integral

Let γ be a Jordan rectifiable curve in $\mathbb{R}^3$. For a continuous function $\Psi : \gamma_\zeta \to \mathbb{A}_3$ of the form (4), where $(x, y, z) \in \gamma$, we define an integral along the curve γ_ζ by the equality

$$\begin{aligned}\int\limits_{\gamma_\zeta} \Psi(\zeta)d\zeta := &\sum_{k=1}^{3} e_k \int\limits_{\gamma} U_k(x,y,z)dx + \sum_{k=1}^{3} e_2e_k \int\limits_{\gamma} U_k(x,y,z)dy \\ &+ \sum_{k=1}^{3} e_3e_k \int\limits_{\gamma} U_k(x,y,z)dz + i\sum_{k=1}^{3} e_k \int\limits_{\gamma} V_k(x,y,z)dx \\ &+ i\sum_{k=1}^{3} e_2e_k \int\limits_{\gamma} V_k(x,y,z)dy + i\sum_{k=1}^{3} e_3e_k \int\limits_{\gamma} V_k(x,y,z)dz,\end{aligned}$$

where

$$d\zeta := dx + e_2dy + e_3dz.$$

If a function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is continuous together with partial derivatives of the first order in a domain Ω_ζ, Σ is a piecewise-smooth surface in Ω, and the edge γ of the surface Σ is a rectifiable Jordan curve, then the following analogue of the Stokes formula is true:

$$\begin{aligned}\int\limits_{\gamma_\zeta} \Phi(\zeta)d\zeta = &\int\limits_{\Sigma_\zeta} \left(\frac{\partial\Phi}{\partial x}e_2 - \frac{\partial\Phi}{\partial y}\right)dxdy + \left(\frac{\partial\Phi}{\partial y}e_3 - \frac{\partial\Phi}{\partial z}e_2\right)dydz \\ &+ \left(\frac{\partial\Phi}{\partial z} - \frac{\partial\Phi}{\partial x}e_3\right)dzdx. \qquad (6)\end{aligned}$$

Now, the next theorem is a result of the formula (6) and the equalities (2).

Theorem 2. *Suppose that $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is a monogenic function in a domain Ω_ζ, Σ is a piecewise-smooth surface in Ω, and the edge γ of the surface Σ is a rectifiable Jordan curve. Then*

$$\int\limits_{\gamma_\zeta} \Phi(\zeta)d\zeta = 0. \qquad (7)$$

Now, similarly to the proof of Theorem 3.2 [4] we can prove the following

Theorem 3. *Let $\Phi : \Omega_\zeta \to \mathbb{A}_3$ be a monogenic function in a domain Ω_ζ. Then for every closed Jordan rectifiable curve γ homotopic to a point in Ω, the equality* (7) *holds.*

For functions taking values in the algebra $\mathbb{A}_3$, the following Morera theorem can be established in the usual way.

Theorem 4. *If a function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is continuous in a domain Ω_ζ and satisfies the equality*

$$\int\limits_{\partial\triangle_\zeta} \Phi(\zeta)d\zeta = 0 \tag{8}$$

for every triangle $\triangle_\zeta$ such that $\overline{\triangle_\zeta} \subset \Omega_\zeta$, then the function Φ is monogenic in the domain Ω_ζ.

4. Cauchy integral formula

In what follows, we consider a harmonic basis $\{e_1, e_2, e_3\}$ with the following decomposition with respect to the basis $\{1, \rho_1, \rho_2\}$:

$$e_1 = 1, \qquad e_2 = i + \frac{1}{2}\, i\rho_2, \qquad e_3 = -\rho_1 - \frac{\sqrt{3}}{2}\, i\rho_2.$$

It follows from Lemma 1.1 [2] that

$$\zeta^{-1} = \frac{1}{x+iy} + \frac{z}{(x+iy)^2}\,\rho_1 + \left(\frac{i}{2}\,\frac{\sqrt{3}z - y}{(x+iy)^2} + \frac{z^2}{(x+iy)^3}\right)\rho_2 \tag{9}$$

for all $\zeta = x + ye_2 + ze_3 \in E_3 \setminus \{ze_3 : z \in \mathbb{R}\}$. Thus, it is obvious that the straight line $\{ze_3 : z \in \mathbb{R}\}$ is contained in the radical of the algebra $\mathbb{A}_3$.

Using the equality (9), it is easy to calculate that

$$\int\limits_{\tilde\gamma_\zeta} \tau^{-1}d\tau = 2\pi i, \tag{10}$$

where $\tilde\gamma_\zeta := \{\tau = x + ye_2 : x^2 + y^2 = R^2\}$.

Theorem 5. *Let Ω be a domain convex in the direction of the axis Oz and $\Phi : \Omega_\zeta \to \mathbb{A}_3$ be a monogenic function in the domain Ω_ζ. Then for every point $\zeta_0 \in \Omega_\zeta$ the following equality is true:*

$$\Phi(\zeta_0) = \frac{1}{2\pi i}\int\limits_{\gamma_\zeta} \Phi(\zeta)\,(\zeta - \zeta_0)^{-1}\,d\zeta, \tag{11}$$

where γ_ζ is an arbitrary closed Jordan rectifiable curve in Ω_ζ, which surrounds once the straight line $\{\zeta_0 + ze_3 : z \in \mathbb{R}\}$.

Proof. We represent the integral on the right-hand side of the equality (11) as the sum of the following two integrals:

$$\int\limits_{\gamma_\zeta} \Phi(\zeta)\,(\zeta-\zeta_0)^{-1}\,d\zeta = \int\limits_{\gamma_\zeta} (\Phi(\zeta)-\Phi(\zeta_0))\,(\zeta-\zeta_0)^{-1}\,d\zeta + \\ +\Phi(\zeta_0)\int\limits_{\gamma_\zeta} (\zeta-\zeta_0)^{-1}\,d\zeta =: I_1+I_2.$$

Inasmuch as the domain Ω is convex in the direction of the axis Oz and the curve γ_ζ surrounds once the straight line $\{\zeta_0 + ze_3 : z\in\mathbb{R}\}$, γ is homotopic to the circle

$$K(R) := \{(x-x_0)^2+(y-y_0)^2 = R^2, z = z_0\},\ \text{where}\ \ \zeta_0 = x_0+y_0e_2+z_0e_3.$$

Then using the equality (10), for $\tau=\zeta-\zeta_0$, we have $I_2 = 2\pi i\Phi(\zeta_0)$.

Let us prove that $I_1 = 0$. First, we choose on the curve γ two points A and B in which there are tangents to γ, and we choose also two points A_1, B_1 on the circle $K(\varepsilon)$ which is completely contained in the domain Ω. Let γ^1, γ^2 be connected components of the set $\gamma\setminus\{A,B\}$. By K^1 and K^2 we denote connected components of the set $K(\varepsilon)\setminus\{A_1,B_1\}$ in such a way that after a choice of smooth arcs Γ^1, Γ^2 each of the closed curves $\gamma^1\cup\Gamma^2\cup K^1\cup\Gamma^1$ and $\gamma^2\cup\Gamma^1\cup K^2\cup\Gamma^2$ will be homotopic to a point of the domain $\Omega\setminus\{(x_0,y_0,z) : z\in\mathbb{R}\}$.

Then it follows from Theorem 3 that

$$\int\limits_{\gamma_\zeta^1\cup\Gamma_\zeta^2\cup K_\zeta^1\cup\Gamma_\zeta^1} (\Phi(\zeta)-\Phi(\zeta_0))\,(\zeta-\zeta_0)^{-1}\,d\zeta = 0, \tag{12}$$

$$\int\limits_{\gamma_\zeta^2\cup\Gamma_\zeta^1\cup K_\zeta^2\cup\Gamma_\zeta^2} (\Phi(\zeta)-\Phi(\zeta_0))\,(\zeta-\zeta_0)^{-1}\,d\zeta = 0. \tag{13}$$

Inasmuch as each of the curves Γ_ζ^1, Γ_ζ^2 has different orientations in the equalities (12), (13), after addition of the mentioned equalities we obtain

$$\int\limits_{\gamma_\zeta} (\Phi(\zeta)-\Phi(\zeta_0))\,(\zeta-\zeta_0)^{-1}\,d\zeta = \int\limits_{K_\zeta(\varepsilon)} (\Phi(\zeta)-\Phi(\zeta_0))\,(\zeta-\zeta_0)^{-1}\,d\zeta, \tag{14}$$

where the curves $K_\zeta(\varepsilon)$ and γ_ζ have the same orientation.

The integrand in the right-hand side of the equality (14) is bounded by a constant which does not depend on ε. Therefore, passing to the limit in the equality (14) as $\varepsilon\to 0$, we obtain $I_1 = 0$ and the theorem is proved.

Using the formula (11), we obtain the Taylor expansion of monogenic function in the usual way. Thus, as in the complex plane, one can give different equivalent definitions of a monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$, i. e. the following theorem is true:

Theorem 6. *Let Ω be a domain convex in the direction of the axis Oz. Then a function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is monogenic in the domain Ω_ζ if and only if one of the following conditions is satisfied:*

(I) *the components* $U_k : \Omega \to \mathbb{C}$, $k = \overline{1,3}$, *of the decomposition*

$$\Phi(\zeta) = \sum_{k=1}^{3} U_k(x, y, z)\ e_k\,,$$

of the function Φ *are differentiable with respect to the variables* x, y, z *in* Ω *and the conditions* (2) *are satisfied in the domain* Ω_ζ*;*

(II) *the function* Φ *is continuous in* Ω_ζ *and satisfies the equality* (8) *for every triangle* $\triangle_\zeta$ *such that* $\overline{\triangle_\zeta} \subset \Omega_\zeta$*;*

(III) *for every* $\zeta_0 \in \Omega_\zeta$ *there exists a neighbourhood in which the function* Φ *is expressed as the sum of the power series*

$$\Phi(\zeta) = \sum_{k=0}^{\infty} c_k\ (\zeta - \zeta_0)^k, \qquad c_k \in \mathbb{A}_3.$$

References

[1] I. P. Mel'nichenko, *Algebras of functionally-invariant solutions of the three-dimensional Laplace equation*, Ukr. Math. J. **55**, no. 9 (2003), 1551–1557.

[2] I. P. Mel'nichenko and S. A. Plaksa, *Commutative algebras and spatial potential fields*, Kiev: Inst. Math. NAS Ukraine, 2008 [in Russian].

[3] E. R. Lorch, *The theory of analytic function in normed abelian vector rings*, Trans. Amer. Math. Soc. **54** (1943), 414–425.

[4] E. K. Blum, *A theory of analytic functions in Banach algebras*, Trans. Amer. Math. Soc. **78** (1955), 343–370.

[5] A. Sudbery, *Quaternionic analysis*, Math. Proc. Camb. Phil. Soc. **85** (1979), 199–225.

[6] F. Colombo, I. Sabadini, and D. Struppa, *Slice monogenic functions*, arXiv:0708.3595v2 [math.CV] 25 Jan 2008.

[7] F. Brackx and R. Delanghe, *Duality in hypercomplex functions theory*, J. Funct. Anal. **37**, no. 2 (1980), 164–181.

[8] S. Bernstein, *Factorization of the nonlinear Schrödinger equation and applications*, Complex Variables and Elliptic Equations **51**, no. 5–6 (2006), 429–452.

[9] V. V. Kravchenko and M. V. Shapiro, *Integral representations for spatial models of mathematical physics*, Pitman Research Notes in Mathematics, Addison Wesley Longman Inc., 1996.

[10] W. Sprössig, *Eigenvalue problems in the framework of Clifford analysis*, Advances in Applied Clifford Algebras **11** (2001), 301–316.

[11] J. Ryan, *Dirac operators, conformal transformations and aspects of classical harmonic analysis*, J. of Lie Theory **8** (1998), 67–82.

[12] W. Sprössig, *Quaternionic analysis and Maxwell's equations*, CUBO, A Math. J. **7**, no. 2 (2005), 57–67.

[13] B. Schneider and E. Karapinar, *A note on biquaternionic MIT bag model*, Int. J. Contemp. Math. Sci. **1**, no. 10 (2006), 449–461.

[14] A. S. Meylekhzon, *On monogeneity of quaternions*, Doklady Acad. Nauk SSSR **59**, no. 3 (1948), 431–434 [in Russian].

Institute of Mathematics
National Academy of Sciences of Ukraine
3 Tereshchenkivs'ka Street
UA-01 601 Kyiv 4, Ukraine
e-mail: shpakivskyi@mail.ru
e-mail: plaksa@imath.kiev.ua

Presented by Andrzej Łuczak at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 26, 2010

TWIERDZENIA CAŁKOWE I WZÓR CAUCHY'EGO W PRZEMIENNEJ TRÓJWYMIAROWEJ ALGEBRZE HARMONICZNEJ

S t r e s z c z e n i e

Dla funkcji jednoznacznie odwracalnych o wartościach w trójwymiarowej przemiennej algebrze harmonicznej z jedynką i dwuwymiarowym pierwiastkiem, dowodzimy odpowiedników klasycznych twierdzeń całkowych teorii funkcji analitycznych jednej zmiennej zespolonej: twierdzeń całkowych Cauchy'ego dla całki powierzchniowej i krzywoliniowej twierdzenia Morery i wzoru całkowego Cauchy'ego.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2010 Vol. LX

Recherches sur les déformations no. 2

pp. 55–63

Dmytro Mierzejewski

THE DIMENSIONS OF SECTIONS OF SOLUTION SETS FOR SOME QUADRATIC QUATERNIONIC EQUATIONS

Summary

We study the sets of the solutions of quadratic quaternionic equations of two certain kinds by the method of sections by hyperplanes perpendicular to the real axis. Namely, we study the question about possible dimensions of such sections. For the both considered kinds of equations we obtain that any considered section cannot be three-dimensional and it can be two-dimensional only in some special cases. In particular, for any quaternionic equation of the form

$$ax^2 + x^2b + xcx + \sum_{\ell=1}^{m} p^{(\ell)} x q^{(\ell)} + s = 0$$

with $c \notin \mathbb{R}$ such section can be two-dimensional only if it contains a plane.

1. Introduction

Investigations of solutions of polynomial quaternionic equations (or, by other words, zeros of quaternionic polynomials) with one unknown were performed in many works during 20th and 21st century, for example [1–11].

It is already well-known that, in contrast to real and complex polynomial equations (with one unknown), it occurs often that a quaternionic one has infinitely many solutions. In particular, the set of the solutions or a part of this set may constitute a sphere, a circle, an unbounded linear manifold, etc. In this paper we will study a question about possible dimensions of these sets for some certain types of equations. More exactly, we will consider dimensions of so-called "real-fixing sections" of sets of solutions (the corresponding definition is in Section 2).

In our investigations we will use the method of passing from a quaternionic equation with one unknown to the corresponding system of four real equations with four unknowns. For this aim one has to rewrite the unknown $x \in \mathbb{H}$ as $x_0 + x_1 i +$

$x_2j + x_3k$ (where $x_0, x_1, x_2, x_3 \in \mathbb{R}$) and analogously every coefficient, to commit multiplications and to get an equation with **real** functions f_0, f_1, f_2, f_3:

$$f_0(x_0, x_1, x_2, x_3) + f_1(x_0, x_1, x_2, x_3)i$$
$$+f_2(x_0, x_1, x_2, x_3)j + f_3(x_0, x_1, x_2, x_3)k = 0.$$

This equation is equivalent to the following system of four real equations:

$$\begin{cases} f_0(x_0, x_1, x_2, x_3) = 0, \\ f_1(x_0, x_1, x_2, x_3) = 0, \\ f_2(x_0, x_1, x_2, x_3) = 0, \\ f_3(x_0, x_1, x_2, x_3) = 0. \end{cases}$$

This method may be named very direct, simple as an idea, and often difficult in applying. Yet, it seems to be almost impossible to avoid this method in the present-day investigations of polynomial quaternionic equations. We have already used it successfully in [3–6].

2. Preliminaries

We use $\mathbb{H}$ as the standard notation for the set of all (real) quaternions. (We do not deal with so-called complex quaternions whose components are complex.) The notation $\mathbb{R}$ has its usual sense.

We use the standard notations i, j, k for the quaternionic imaginary units; recall that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

We always use lower indices to denote the components of a quaternion as follows:

$$\mathbb{H} \ni \xi = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k,$$

where $\xi_0, \xi_1, \xi_2, \xi_3 \in \mathbb{R}$.

We often treat sets of some quaternions geometrically using the well-known interpretation of any quaternion $\xi = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$ as the point $(\xi_0, \xi_1, \xi_2, \xi_3)$ of the four-dimensional space.

We name a *quaternionic equation* any one in which every known parameter is a quaternion; as for solutions of such equations, we always consider solutions being quaternions. Analogously, *real equations* are ones with real parameters and real solutions being considered. In every quaternionic equation of this paper the letter x denotes the unknown and other letters denote given parameters, if there is no other explanation.

Let S be the set of the solutions of a given polynomial quaternionic equation. Fix a certain real number ξ_0, and let S_{ξ_0} be the set of such $x \in S$ for which $x_0 = \xi_0$. We will call this set S_{ξ_0} a *real-fixing section* of the set of the solutions of the given equation. Geometrically S_{ξ_0} is the intersection of S and the (three-dimensional) hyperplane given by $x_0 = \xi_0$ (where an arbitrary point of the hyperplane is denoted by (x_0, x_1, x_2, x_3)). Thus S_{ξ_0} is really a "section" of S by this hyperplane if

one agrees to use the word "section" for such three-dimensional structure. Such real-fixing section can be viewed as a figure in the usual three-dimensional space of points (x_1, x_2, x_3).

In what follows the notation S_{ξ_0} will be used sometimes without additional explanation in the same sense as in the previous paragraph.

3. Simple quadratic equations with a split square

We will call a quadratic quaternionic equation *simple* if every its quadratic term has only one constant, that is, can be written as either αx^2, or $x^2\alpha$, or $x\alpha x$. So, an arbitrary simple quadratic quaternionic equation can be written as

$$\sum_{\ell=1}^{\lambda} a^{(\ell)}x^2 + \sum_{\ell=1}^{\mu} x^2 b^{(\ell)} + \sum_{\ell=1}^{\nu} xc^{(\ell)}x + \sum_{\ell=1}^{m} p^{(\ell)}xq^{(\ell)} + s = 0.$$

Yet, a very useful property of such simple equations is the fact that now it is possible to apply distributivity to decrease the number of the terms. As a result an arbitrary simple quadratic quaternionic equation can be rewritten as

$$ax^2 + x^2 b + xcx + (pxq) + s = 0, \text{ where } (pxq) = \sum_{\ell=1}^{m} p^{(\ell)}xq^{(\ell)}. \tag{1}$$

Now we will study the dimensions of real-fixing sections of the sets of the solutions of equations of this form.

Note also that we call the terms ax^2 and $x^2 b$ *non-split squares*, while the term xcx is called a *split square*. (In general, that is, if an equation may be not simple, a split square has the form $\alpha x\beta x\gamma$ and a non-split one has the form $\alpha x^2\gamma$. Such terminology will be used in Section 4.) If the coefficient of at least one from these three terms is real then it is possible to change position of the coefficient; for example, if $c \in \mathbb{R}$ then one can make a non-split square cx^2 from the split square xcx. For the theorem below the presence of a "true" split square will be essential, and thus we will assume that $c \notin \mathbb{R}$.

Theorem 1. *Let a quaternionic equation of the form* (1) *be given, where* $c \notin \mathbb{R}$. *Then any real-fixing section of the set of the solutions of this equation is either the empty set, or a d-dimensional set with* $d \leq 1$, *or a two-dimensional set containing a plane.*

Proof. Let us pass from (1) to the corresponding system of real equations. It will be unnecessary for our considerations to write down carefully terms generated by (pxq). It is obvious that these terms constitute an expression linear with respect to the unknowns in each equation of the system. Thus we will denote these linear expressions by $L_t(x_0, x_1, x_2, x_3)$, where t is an integer indicating which equation of the system is considered.

So, direct calculations give the following system equivalent to (1):

$$
(2)\qquad \begin{cases}
-(a_0+b_0+c_0)x_1^2-(a_0+b_0+c_0)x_2^2-(a_0+b_0+c_0)x_3^2 \\
\quad -2(a_1+b_1+c_1)x_0x_1-2(a_2+b_2+c_2)x_0x_2-2(a_3+b_3+c_3)x_0x_3 \\
\quad +L_0(x_0,x_1,x_2,x_3)+(a_0+b_0+c_0)x_0^2+s_0=0, \\
-(a_1+b_1+c_1)x_1^2-(a_1+b_1-c_1)x_2^2-(a_1+b_1-c_1)x_3^2 \\
\quad -2c_2x_1x_2-2c_3x_1x_3+2(a_0+b_0+c_0)x_0x_1+2(b_3-a_3)x_0x_2 \\
\quad +2(a_2-b_2)x_0x_3+L_1(x_0,x_1,x_2,x_3)+(a_1+b_1+c_1)x_0^2+s_1=0, \\
-(a_2+b_2-c_2)x_1^2-(a_2+b_2+c_2)x_2^2-(a_2+b_2-c_2)x_3^2 \\
\quad -2c_1x_1x_2-2c_3x_2x_3+2(a_3-b_3)x_0x_1+2(a_0+b_0+c_0)x_0x_2 \\
\quad +2(b_1-a_1)x_0x_3+L_2(x_0,x_1,x_2,x_3)+(a_2+b_2+c_2)x_0^2+s_2=0, \\
-(a_3+b_3-c_3)x_1^2-(a_3+b_3-c_3)x_2^2-(a_3+b_3+c_3)x_3^2 \\
\quad -2c_1x_1x_3-2c_2x_2x_3+2(b_2-a_2)x_0x_1+2(a_1-b_1)x_0x_2 \\
\quad +2(a_0+b_0+c_0)x_0x_3+L_3(x_0,x_1,x_2,x_3) \\
\quad +(a_3+b_3+c_3)x_0^2+s_3=0.
\end{cases}
$$

We have to prove that with each fixed x_0 the dimension of the set of the solutions of this system does not equal 3 and if it equals 2 then it contains a plane.

Each equation of (2) is an equation of degree 2 with respect to $x_1,\ x_2,\ x_3$, so that as a rule it generates a two-dimensional quadric; a three-dimensional set appears only if every coefficient of the equation equals 0, and then it is the whole three-dimensional space. We have to consider the intersection of four sets generated by all equations of the system. Obviously, this intersection may be tree-dimensional only if every equation generates a three-dimensional set, that is, has only zero coefficients. It is easy to see that in this case $c_1=c_2=c_3=0$, so that $c\in\mathbb{R}$; it is in contradiction with the condition of the theorem. So, we have already proved that the dimension of the set of the solutions does not equal 3.

Then we investigate the case where the dimension equals 2. First of all, take into attention the following known geometric fact: if the intersection of two two-dimensional quadrics is a two-dimensional figure then either these quadrics coincide, or their intersection contains a plane (in the last case each quadric is the union of two planes (distinct or coinciding), and the intersection may be a plane or the union of a plane and a line). Since a plane in the intersection is allowed by the theorem, we should consider (and refute) only the case where the intersection would be two-dimensional, but containing no plane. It means that two-dimensional quadrics can provide such intersection only if they coincide.

Taking into attention the information from the previous paragraph, it is easy to see that we should consider the case where one of the following four situations takes place:

1) every equation of (2) generates **the same** two-dimensional quadric;

2) three equations of (2) generate **the same** two-dimensional quadric, and one equation of (2) generates the whole three-dimensional space;

3) two equations of (2) generate **the same** two-dimensional quadric, and two equations of (2) generate the whole three-dimensional space;

4) one equation of (2) generates a two-dimensional quadric, and three equations of (2) generate the whole three-dimensional space.

We stress that the words "the same" hereafter mean "having completely identical position in the space" (not only shape and size).

Is it possible for the second equation to generate the three-dimensional space? Supposing that all its coefficients are zeros we get in particular:

$$a_1 + b_1 + c_1 = a_1 + b_1 - c_1 = c_2 = c_3 = 0.$$

These equalities imply that $c_1 = c_2 = c_3 = 0$ and thus $c \in \mathbb{R}$, a contradiction with the condition of the theorem. Therefore it is impossible for the second equation to generate the three-dimensional space. Quite analogously one proves that neither the third equation, nor the fourth one generates the three-dimensional space. Thus only the first equation may do, and from the above-mentioned four situations only the first two are possible.

Suppose that every equation of (2) generates the same two-dimensional quadric (the first situation). Then the first equation may generate only a sphere (since the coefficients at x_1^2, x_2^2, and x_3^2 are the same and there is no term with x_1x_2, x_1x_3, or x_2x_3). Since all four quadrics have to be the same, it is necessary that a sphere should be in every other equation. Then these three equations have to possess the same property as was just mentioned in brackets. Thus it is easy to see that $c_1 = c_2 = c_3 = 0$, and this gives again a contradiction.

So, the only possibility is the second situation. Moreover it is understandable from above reasonings that the equation generating the whole three-dimensional space is the first one. The other three equations have to generate the same two-dimensional quadric. But it is possible only if the coefficients of the equations are proportional (see, for example, [12]); in particular, if one equation has zero at a certain product of the unknowns then other equations have zero at the same product. Applying this to the products x_1x_2, x_1x_3, and x_2x_3 we easily get

$$c_1 = c_2 = c_3 = 0,$$

and this gives again a contradiction.

So, any situation giving a two-dimensional surface in the intersection is impossible if this surface does not contain any plane, and the theorem is proved. □

It is also interesting to note that both 0- and 1-dimensional sets, allowed according to Theorem 1, as far as the empty set, are really present among real-fixing sections of the sets of the solutions of equations of the form (1). Moreover there exists at least one equation of this form whose set of solutions has real-fixing sections of all these three types. An example is

$$xix = i.$$

Solving it by passing to a system of real equations it is not difficult to obtain its solution in the form

$$\begin{cases} x_1 = 0, \\ x_2^2 + x_3^2 = x_0^2 - 1. \end{cases}$$

If $x_0 < 1$ then the real-fixing section is the empty set. If $x_0 = 1$ then the real-fixing section is a point, that is, its dimension is 0. If $x_0 > 1$ then the real-fixing section is a circle in the plane $x_1 = 0$, that is, its dimension is 1.

As for figures containing planes, we do not know whether they can arise as real-fixing sections of the set of solutions of (1).

4. A type of quadratic equations with two split squares

Let us pass to quadratic quaternionic equations containing two split squares. In this section we will study a very narrow particular case of such equations. Note firstly that the considered equations will have no other terms besides two split squares and a constant:

$$\alpha x \beta x \gamma + \lambda x \mu x \nu + \sigma = 0. \tag{3}$$

In addition we assume that there is no zero among the coefficients α, β, γ, λ, μ, and ν (to assure that any from the both split squares does not equal 0). The equation (3) can be simplified by multiplication by γ^{-1} on the right and by λ^{-1} on the left. Then it gets the following form:

$$axbx + xcxd + s = 0. \tag{4}$$

At last we introduce one more restriction on the coefficients of (4). Namely, we assume that the quaternions a, b, c, and d have such a simple structure that all their real components and components at j and k equal 0, so that

$$a = a_1 i,\ b = b_1 i,\ c = c_1 i,\ d = d_1 i \quad \text{where} \quad a_1, b_1, c_1, d_1 \in \mathbb{R}. \tag{5}$$

The following theorem gives some information about dimensions of real-fixing sections of the set of the solutions of such equations.

Theorem 2. *Let a quaternionic equation of the form* (4) *be given with non-zero coefficients* a, b, c, d *of the form* (5). *Then any real-fixing section of the set of the solutions of this equation is not of the dimension* 3. *Moreover if some* S_{ξ_0} *of these real-fixing sections is of dimension* 2 *then one of the following two situations occurs:*

1) *the section contains a plane;*

2) $\xi_0 = 0$ *and this section is a hyperboloid or a conic surface.*

Proof. Taking into attention the structure of the coefficients of the given equation one obtains the following equivalent system of real equations:

$$\begin{cases} (a_1b_1 + c_1d_1)x_1^2 - (a_1b_1 + c_1d_1)x_2^2 - (a_1b_1 + c_1d_1)x_3^2 \\ \quad -(a_1b_1 + c_1d_1)x_0^2 + s_0 = 0, \\ -2(a_1b_1 + c_1d_1)x_0x_1 + s_1 = 0, \\ 2(a_1b_1 - c_1d_1)x_1x_3 + s_2 = 0, \\ 2(a_1b_1 - c_1d_1)x_1x_2 + s_3 = 0. \end{cases} \tag{6}$$

Let us introduce notations A and B for the following real numbers:

$$A := a_1b_1 + c_1d_1, \qquad B := a_1b_1 - c_1d_1.$$

Then we can rewrite (6) as follows:

$$(7) \qquad \begin{cases} Ax_1^2 - Ax_2^2 - Ax_3^2 - Ax_0^2 + s_0 = 0, \\ -2Ax_0x_1 + s_1 = 0, \\ 2Bx_1x_3 + s_2 = 0, \\ 2Bx_1x_2 + s_3 = 0. \end{cases}$$

We will investigate separately the following cases:

1) $A = B = 0$;
2) $A = 0,\ B \neq 0$;
3) $A \neq 0$.

First of all, it is easy to conclude that the first case is impossible because in this case $a_1b_1 = c_1d_1 = 0$, that is in contradiction with the condition of the theorem.

Then let $A = 0,\ B \neq 0$. In this case we can rewrite (7) as follows:

$$\begin{cases} s_0 = 0, \\ s_1 = 0, \\ 2Bx_1x_3 + s_2 = 0, \\ 2Bx_1x_2 + s_3 = 0. \end{cases}$$

Here all real-fixing sections are identical (since x_0 is absent in the system) and they are not empty only if $s_0 = s_1 = 0$; then it is sufficient to consider the following system:

$$\begin{cases} 2Bx_1x_3 + s_2 = 0, \\ 2Bx_1x_2 + s_3 = 0. \end{cases}$$

Since $B \neq 0$, the dimension cannot equal 3. Moreover these two equations cannot generate the same quadric (since the first one has 0 at x_1x_2 and $2B \neq 0$ at x_1x_3, but the second one has $2B \neq 0$ at x_1x_2 and 0 at x_1x_3). Similarly to analogous reasoning in the proof of Theorem 1 we conclude that if the intersection of these two quadrics is two-dimensional, then it contains a plane and thus the theorem holds in this case.

Let at last $A \neq 0$. Then the first equation of (7) generates a hyperboloid or a conic surface. It is clear that any other equation of this system cannot generate the same surface as the first one. Therefore: 1) any three-dimensional set in the intersection is impossible; 2) one can expect to get a two-dimensional one only if each from the last three equations of (7) generates the whole three-dimensional space (of course, with some fixed x_0). This is possible only if all coefficients of these equations are zeros. Then, observing the second equation, we conclude in particular that $-2Ax_0 = 0$. Since now $A \neq 0$, we get $x_0 = 0$. The fixed x_0 is denoted by ξ_0 in the theorem. So, it is understandable that the theorem is again true (and, in particular, now the section is a hyperboloid or a conic surface). □

It is interesting to consider examples of equations of the form (4) providing hyperboloids or conic surfaces in the real-fixing sections S_0. An example is

$$ixix + xixi + s_0 = 0, \tag{8}$$

where $s_0 \in \mathbb{R}$ (s_0 has to equal 0 for a conic surface and some other real number for a hyperboloid). According to the terminology of [4], the equation (8) is **not** multi-quasi-spherical. Thus now we have refuted a hypothesis stated in [4] and [5] that all quadratic quaternionic equations would be multi-quasi-spherical.

Remark. Performing some reasonings of symmetry it is easy to understand that Theorem 2 remains true if the conditions (5) are changed toy

$$a = a_1 j,\ b = b_1 j,\ c = c_1 j,\ d = d_1 j$$

or

$$a = a_1 k,\ b = b_1 k,\ c = c_1 k,\ d = d_1 k.$$

Acknowledgements

The author is thankful to several participants of the internet-forum *dxdy.ru* (namely, those whose nicknames are "Mathusic", "Aleksey K.", "Professor Snape", and "Someone") for useful discussions on some geometric items.

References

[1] S. Eilenberg and I. Niven, *The "fundamental theorem of algebra" for quaternions*, Bulletin of the American Mathematical Society **50**, no. 4 (1944), 246–248.

[2] D. Janovská and G. Opfer, *Linear equations in quaternionic variables*, Mitt. Math. Ges. Hamburg **27** (2008), 223–234.

[3] D. Mierzejewski, *Investigation of quaternionic quadratic equations I. Factorization and passing to a system of real equations*, Bull. Soc. Sci. Lettres Łódź **58** Sér. Rech. Déform. **56** (2008), 17–26.

[4] D. Mierzejewski, *Quasi-spherical and multi-quasi-spherical polynomial quaternionic equations: introduction of the notions and some examples*, Advances in Applied Clifford Algebras, submitted.

[5] D. Mierzejewski, *Linear manifolds in sets of solutions of quaternionic polynomial equations of several types*, Advances in Applied Clifford Algebras, submitted.

[6] D. Mierzejewski, *Spheres in sets of solutions of quadratic quaternionic equations of some types*, Bull. Soc. Sci. Lettres Łódź **60** Sér. Rech. Déform. **60**, no. 1 (2010) p. 33–43.

[7] D. Mierzejewski and V. Szpakowski, *On solutions of some types of quaternionic quadratic equations*, Bull. Soc. Sci. Lettres Łódź **58** Sér. Rech. Déform. **55** (2008), 49–58.

[8] I. Niven, *The roots of a quaternion*, American Mathematical Monthly **49** (1942), 386–388.

[9] A. Pogorui and M. Shapiro, *On the structure of the set of the zeros of quaternionic polynomials*, Complex Variables and Elliptic Equations **49**, no. 6 (2004), 379–389.

[10] V. Szpakowski, *Solution of general linear quaternionic equations*, The XI Kravchuk International Scientific Conference, Kyiv (Kiev), Ukraine, 2006, p. 661 (in Ukrainian).

[11] V. Szpakowski, *Solution of quadratic quaternionic equations*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform., to appear.

[12] E. Vinberg, *The Course of Algebra*, Factorial Press, Moscow 2001, 544 pp. (in Russian).

Teatralna Street, 5-b, flat 6
Zhytomyr, 10014
Ukraine
e-mail: dmytro1972@gmail.com

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on July 16, 2010

WYMIARY PRZEKROJÓW ZBIORÓW ROZWIĄZAŃ NIEKTÓRYCH RÓWNAŃ KWADRATOWYCH KWATERNIONOWYCH

S t r e s z c z e n i e

Badamy zbiory rozwiązań równań kwadratowych kwaternionowych pewnych dwóch rodzajów metodą przekrojów hiperpłaszczyznami prostopadłymi do osi rzeczywistej. Mianowicie, uzyskujemy odpowiedź na pytanie o możliwe wymiary takich przekrojów. Dla obu rozpatrywanych rodzajów równań otrzymujemy, że żaden z rozpatrywanych przekrojów nie może być trójwymiarowy, lecz dwuwymiarowy i to tylko w niektórych szczególnych przypadkach, między innymi, dla dowolnego równania kwaternionowego postaci

$$ax^2 + x^2b + xcx + \sum_{\ell=1}^{m} p^{(\ell)} x q^{(\ell)} + s = 0,$$

gdzie $c \notin \mathbb{R}$; taki przekrój może być dwuwymiarowy tylko wtedy, gdy zawiera płaszczyznę.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2010 Vol. LX

Recherches sur les déformations no. 2

pp. 65–71

Andrzej Krzysztof Kwaśniewski

SOME COBWEB POSETS DIGRAPHS – ELEMENTARY PROPERTIES AND QUESTIONS

Summary

A digraph that represents reasonably a scheduling problem should have no cycles i.e. it should be DAG i.e. a directed acyclic graph. Here down we shall deal with special kind of graded DAGs named *KoDAGs*. For their definition and first primary properties see [1], where natural join of di-bigraphs (directed bi-parted graphs) and their corresponding adjacency matrices is defined and then applied to investigate cobweb posets and their $Hasse$ digraphs called *KoDAGs*.

In this paper we extend the notion of cobweb poset while delivering some elementary consequences of the description and observations established in [1].

1. Introduction to the subject

It is now a Wiki important knowledge that an incidence structure is a triple $C = (P, L, I)$ where P is a set of *points*, L is a set of *lines* and $I \subseteq P \times L$ is the incidence relation; $I = P \times L$ for *KoDAGs*. (Compare: $V = P \cup L, P \cap L = \emptyset$; $P =$ black vertices = points, $L=$ white vertices=lines).

The elements of I are called *flags*. If $(p, l) \in I$ we say that point p "lies on" line l. The relation I is equivalently defined by its bipartite digraph $G(I)$. The relation I and its bipartite digraph $G(I)$ are equivalently defined by theirs biadjacency matrix. The example of thus efficiently coded finite geometries include such popular examples as Fano plane – a coding potrait of the distinguished composition algebra of J. T. Graves octonions (1843), a friend of William Hamilton, who called them octaves [2].

The *incidence matrix* of an incidence structure C is a **biadjacency matrix** of the Levi graph of the C structure.

The biadjacency matrix of a finite bipartite graph G with n black vertices and m white vertices is an $n \times m$ matrix where the entry a_{ij} is the number of edges joining

black vertex i and white vertex j. In the special case of a finite, undirected, simple bipartite graph, the biadjacency matrix is a Boolean $(0,1)$-matrix.

The adjacency matrix A of a bipartite graph with the **reduced adjacency matrix** or – under synonymous substitution – the **biadjacency Boolean matrix** B is given by

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

The adjacency matrix A of a bipartite **di**graph $\overrightarrow{K}_{k,l}$ (see: [1]) coded via its reduced adjacency or biadjacency Boolean matrix B is according to [1] defined by

$$A = \begin{pmatrix} 0_{k,k} & B(k \times l) \\ 0_{l,k} & 0_{l,l} \end{pmatrix}, \quad \text{where} \quad k = |P|,\ l = |L|.$$

Example 1.

Fig. 1:

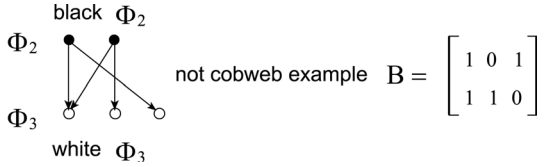


Fig. 2:

Fig. 1 displays (upside down way with respect to drawings in [1]) the bipartite digraph $\overrightarrow{K}_{2,3}$. It is obviously Ferrers dim 1 digraph [1]. Fig. 2 displays the bipartite sub-digraph of the K-digraph $\overrightarrow{K}_{2,3}$. It is not Ferrers dim 1 digraph. What is its Ferrers dimension? Adjoin minimal number of arcs in the Fig. 2 in order to get Ferrers dim 1 digraph, bi-partite, of course.

The adjacency matrices coding digraphs from the example above are shown below.

$$A_{KoDAG} = \begin{bmatrix} O_{2\times 2} & I(2\times 3) \\ O_{3\times 2} & O_{3\times 3} \end{bmatrix}, \quad i.e. \quad A_{KoDAG} = \begin{pmatrix} 0_{2,2} & \begin{smallmatrix}111\\111\end{smallmatrix} \\ 0_{3,3} & 0_{3,3} \end{pmatrix},$$

$$A_{not-cobweb} = \begin{pmatrix} 0_{2,2} & \begin{smallmatrix}101\\110\end{smallmatrix} \\ 0_{3,3} & 0_{3,3} \end{pmatrix},$$

where $O_{s\times s}$ stays for $(k \times m)$ zero matrix while $I(s \times k)$ stays for $(s \times k)$ matrix of ones, i.e. $[I(s \times k)]_{ij} = 1$; $1 \leq i \leq s$, $1 \leq j \leq k$.

Here above in the Example 1 we are led implicitly to the notion of an extended cobweb poset as compared to [1] and references therein. For associated poset – see [1].

Definition 1 (extended cobweb poset – naturally graded). Let $D = (\Phi, \prec \cdot)$ be a transitive irreducible digraph. Let $n \in N \cup \{\infty\}$. Let D be a natural join $D = \oplus{\rightarrow}_{k=0}^{n} B_k$ of Ferrers dim 1 bi-partite subdigraphs B_k of di-bicliques

$$\overrightarrow{K_{k,k+1}} = (\Phi_k \cup \Phi_{k+1}, \Phi_k \times \Phi_{k+1}), n \in N \cup \{\infty\}.$$

The poset $\Pi(D)$ associated to this graded digraph $D = (\Phi, \prec \cdot)$ is called the extended cobweb poset or just *cobweb*, as a colloquial abbreviation.

Sometimes when we are in need we shall distinguish by name the **complete** cobwebs (i.e. cobwebs represented by *KoDAGs*) from the overall family of cobwebs (the extended cobweb posets as introduced above).

Colligate with Levi graph of an incidence structure. Each incidence structure C corresponds to a bipartite graph called Levi graph or incidence graph with a given black and white vertex coloring where black vertices correspond to points and white vertices correspond to lines of C and the edges correspond to flags.

Question 1. Is the natural join operation technique as started in [1] applicable to *sequences of Levi graphs* of incidence structures? The answer is of course in affirmative.

In the case of graded digraphs with the finite set of minimal elements we have what follows (Observation 7 in [1]).

Observation 1. *Consider bipartite digraphs chain obtained from the di-biqliqes chain via deleting or no arcs making thus* [*if deleting arcs*] *some or all of the di-bicliques* $\overrightarrow{K_{k,k+1}}$ *not di-biqliques; denote them as* G_k. *Let* $B_k = B(G_k)$ *denotes their biadjacency Boolean matrices correspondingly. Then for any such* F*-denominated chain* [*hence any chain*] *of bipartite digraphs* G_k *the general formula is:*

$$B\left(\oplus{\rightarrow}_{i=1}^{n} G_i\right) \equiv B[\oplus{\rightarrow}_{i=1}^{n} A(G_i)] = \oplus_{i=1}^{n} B[A(G_i)] \equiv \mathrm{diag}(B_1, B_2, ..., B_n) =$$

$$= \begin{bmatrix} B_1 & & & & \\ & B_2 & & & \\ & & B_3 & & \\ & & & ... & \\ & & & & B_n \end{bmatrix}, \quad n \in N \cup \{\infty\}$$

Comment 1 (not only notation matter). Let us denote by $\langle \Phi_k \rightarrow \Phi_{k+1} \rangle$ the di-bicliques denominated by subsequent levels Φ_k, Φ_{k+1} of the graded F-poset $P(D) = (\Phi, \leq)$ i.e. levels Φ_k, Φ_{k+1} of its cover relation graded digraph $D = (\Phi, \prec \cdot)$ i.e. Hasse diagram (see notation in the author's and others papers quoted in [1]). Then one may

conditionally approve the following identification if necessary natural join condition [1] is implicit within this identification:

$$B\left(\oplus\!\!\rightarrow_{k=1}^{n}\langle\Phi_k\rightarrow\Phi_{k+1}\rangle\right)=B\left(\bigcup_{k=1}^{n}\langle\Phi_k\rightarrow\Phi_{k+1}\rangle\right).$$

If the conditioned set sum of digraphs concerns an ordered digraphs pair satisfying the natural join condition [1] what makes such a conditioned set sum of vertices and simultaneously the set sum of disjoint arcs E_k, E_{k+1} families non-commutative. Note that this what just has been said is exactly the reason of

$$B(G_1\oplus\!\!\rightarrow G_2)=B(G_1\cup G_2)=B(G_1)\oplus B(G_2).$$

2. On number of finite cobwebs an related questions

2.1. Two schemes and a question

Before we deal with questions "how many" let us jot first two schemes of two statements which may be simultaneously referred to relations, their digraphs or corresponding adjacency matrices. Secondly comes an elementary question without giving an answer.

$$\textbf{(Ferrers dim 1)}\oplus\!\!\rightarrow\textbf{(Ferrers dim 1)}=\textbf{(Ferrers dim 1)}.$$

(Obvious: use 2×2 permutation sub-matrix forbidding i.e. 2×2 permutation submatrix disqualification criterion)

$$\textbf{Ferrers}\oplus\!\!\rightarrow\textbf{Ferrers}=\textbf{Ferrers}.$$

See Observation 3 in [1] and note that resulting biadjacency matrices contain none of two 2×2 permutation matrices. *Nota bene* the Observation 3 from [1] follows from the above obvious statements.

Question 2. For biadjacency matrices $B(G_1)=B_1$ and $B(G_2)=B_2$ of bipartite digraphs G_1 and G_2 we have the matrix exponential rule

$$\exp[B_1\oplus B_2]=\exp[B_1]\otimes\exp[B_2]\,,$$

where $\otimes$ stays for the Kronecker product.

Let F be any natural number-valued sequence. Let $\mathbf{A}_F$ denotes the Hasse matrix of the F-denominated cobweb poset $\langle\Phi,\leq\rangle$ [1]. The ζ matrix is then the geometrical series in $\mathbf{A}_F$: $\zeta=(1-\mathbf{A}_F)^{-1\copyright}$. (Recommended: consult the remark from page 12 in [1] on $\zeta=exp[\mathbf{A}]$ in the cases of the Boolean poset 2^N and the Ferrand-Zeckendorf poset of finite subsets of $F=N$ without two consecutive elements [3]). The **Question 2** is: find the rule if any for

$$\zeta_{B_1\oplus B_2}=(1-B_1\oplus B_2)^{-1\copyright}=?$$

2.2. How many

Notation for this subsection
Consider positive integer N composition

$$N = f_1 + f_2 + ... + f_k \quad \text{where} \quad 0 < f_1, f_2, ..., f_k \leq N.$$

The compositions' **type** $\vec{k} = \langle f_1, f_2, ..., f_k \rangle$ labels compositions of the chosen positive integer N, where $N = |V|$ labels on its own the partial graded orders $P_N = \langle V, \leq \rangle$ with N points (vertices) and the partition

$$V = \bigcup_{r=1}^{k} V_r,\ V_r \cap V_s = \oslash \text{ for } r \neq s,\ f_r = |V_r| \text{ and } r, s = 1, ..., k,\ k = 1, ..., N.$$

The partial order $\leq$ is the subset according to $\leq \subseteq V_1 \times V_2 \times ... \times V_k$. The symbol $\left\{ {N \atop k} \right\}$ denotes the array of Stirling numbers of the second kind.

Obvious from obvious and questions
The number of all k-tuples for any k-block ordered partition $< V_1, V_2, \ldots, V_k >$ equals to

$$|V_1| \cdot |V_2| \cdot \ldots \cdot |V_k| = \prod_{r=1}^{k} V_r.$$

The number of all complete cobweb posets $P_N = \langle V, \leq \rangle$ with $|V| = N$ elements is equal to $T_N =$ the number of ordered partitions of V. – Why? Note: The number of ordered partitions of

$$\langle f_1, f_2, ..., f_k \rangle = \vec{k} \quad \text{type is equal to} \quad \binom{n}{f_1, f_2, ..., f_k} = \frac{n!}{f_1! f_2! ... f_k!}.$$

Thereby: the number of all complete cobweb posets

$$P_N = \langle V, \leq \rangle \quad \text{of} \quad \langle f_1, f_2, ..., f_k \rangle = \vec{k} \quad \text{type is equal to} \quad \binom{n}{f_1, f_2, ..., f_k},$$

where $f_r = |V_r|$ and $r = 1, ..., k$ for all particular $\vec{k}$ type k-block ordered partitions

$$\bigcup_{r=1}^{k} V_r = V.$$

Altogether:

2.2.1. The number $\mathrm{Cob}^c(N, \vec{k})$ of all complete of the type $\langle f_1, f_2, ..., f_k \rangle \equiv \vec{k}$ cobweb posets P_N is given by:

$$\mathrm{Cob}^c(N, \vec{k}) = \binom{N}{f_1, f_2, ..., f_k}, \quad N = f_1 + f_2 + ... + f_k, \quad 0 < f_1, f_2, ..., f_k \leq N,$$

2.2.2. The number $Cob^c(N, k)$ of all complete **k**-level cobweb posets P_N reads:

$$\mathrm{Cob}^c(N, k) = \sum_{\substack{f_1 + f_2 + \ldots + f_k = N \\ 0 < f_1, \ldots, f_k \leq N}} \binom{N}{f_1, f_2, ..., f_k} = k! \left\{ {N \atop k} \right\} = \sum_{r=0}^{k} (-1)^{N-k} r^N \binom{N}{r}.$$

$$\mathrm{Cob}^c(N,k) = \quad \text{number of surjections} \quad f : V \mapsto [k].$$

2.2.3. The number $\mathrm{Cob}^c(N)$ of all complete cobweb posets P_N: is then the sum:

$$\mathrm{Cob}^c(N) = \sum_{k=1}^{N} k! \left\{ \begin{array}{c} N \\ k \end{array} \right\}.$$

$$\mathrm{Cob}^c(N) = T_N = \quad \text{the number of } \textbf{ordered} \text{ partitions of } \quad V.$$

2.2.4. The number $K(N, \vec{k})$ of all $\boldsymbol{k}$-ary relations of the given $\vec{k}$ type is:

$$K(N, \vec{k}) = 2^{\prod_{r=1}^{k} V_r} - 1,$$

where

$$N = f_1 + f_2 + ... + f_k, \quad 0 < f_1, f_2, ..., f_k \leq N, \quad f_r = |V_r|,$$

and $r = 1, ..., k$ while $k = 1, ..., N$.

2.2.5. For the number $K(N)$ of all type k-ary relations , $k = 1, \ldots, N$ we then have

$$K(N) = \sum_{\substack{f_1+f_2+\ldots+f_k=N \\ 0<f_1,\ldots,f_k\leq N}} [2^{f_1 \cdot f_2 \cdot \ldots \cdot f_k} - 1].$$

2.2.6. Question 3. The number of all $\boldsymbol{k}$-level graded posets $P_N = \langle V, \leq \rangle$ with $|V| = N$ elements where the partial order $\leq$ is the subset of Cartesian product:

$$\leq \subseteq V_1 \times V_2 \times ... \times V_k$$

and where

$$f_r = |V_r|, \quad r = 1, ..., k \quad \text{and} \quad k = 1, ..., N$$

while $N = f_1 + f_2 + ... + f_k$, $0 < f_1, f_2, ..., f_k \leq N$... **equals ?**

2.2.7. Question 4. The number of all graded posets $P_N = \langle V, \leq \rangle$ with $|V| = N$ elements for any type $\vec{k}$, $k = 1, \ldots, N$... **equals ?**

Acknowledgments

Let me thank the Student of Gdańsk University Maciej Dziemiańczuk for his alert assistance. I am indebted with gratitude to my wife for being so patient with me still browsing cobwebs in spite of rolling around Christmas Eve.

References

[1] A. K. Kwaśniewski, *Cobweb posets and KoDAG digraphs are representing natural join of relations, their di-bigraphs and the corresponding adjacency matrices*, arXiv:math/0812.4066v1,[v1] Sun, 21 Dec 2008 23:04:48 GMT.

[2] A. K. Kwaśniewski, *Glimpses of the Octonions and Quaternions History and Todays Applications in Quantum Physics*, arXiv:math/0803.0119v1 ,[v1] Sun, 2 Mar 2008 12:39:16 GMT.

[3] E. Ferrand, *An analogue of the Thue-Morse sequence*, The Electronic Journal of Combinatorics **14** (2007), 30.

Institute of Combinatorics and its Applications
High School of Mathematics and Applied Informatics
Kamienna 17, PL-15-021 Białystok
Poland
e-mail: kwandr@gmail.com

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on July 16, 2010

SKIEROWANE GRAFY TZW. COBWEB POSETÓW – KILKA WŁAŚCIWOŚCI ORAZ ZAGADNIEŃ ELEMENTARNYCH

S t r e s z c z e n i e

Grafy skierowane opisujące rozsądne algorytmy zadań – to grafy acykliczne czyli tzw. DAG's, które można zawsze odczytywać jako diagramy Hasse'go częściowo uporządkowanych zbiorów (posets). W niniejszym artykule rozważa się szczególne częściowo uporządkowane zbiory ze stopniowaniem zwane "cobweb posets". Stanowią one w złączaniu naturalnym ciągi Kompletnych Grafów dwudzielnych – uporządkowanych (ordered) oraz skierowanych i acyklicznych (DAG's). Na cześć Profesora Kazimierza Kuratowskiego – współtwórcy współczesnej teorii grafów – autor nazwał owe grafy Hassego – *KoDAGs*.

W niniejszej pracy przedstawia się elementarne skutki takiego określenia struktury *KoDAG* oraz zadaje się – wraz ze wskazówkami – kilka naturalnych zadań-ćwiczeń o charakterze kombinatorycznym.

PL ISSN 0459-6854

BULLETIN

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2010 Vol. LX

Recherches sur les déformations no. 2

pp. 73–92

Małgorzata Nowak-Kępczyk

BINARY ALLOY THIN FILMS VS. LENNARD-JONES AND MORSE POTENTIALS

NOTE ON BINARY ALLOYS WITH ARBITRARY ATOMS CONCENTRATIONS

Summary

The study of thin films has been very intense during the last decades. It is observed, both theoretically and experimentally that in thin films samples the concentrations of atoms differ between inner and outer layers in broad range of temperature. In this paper we shall present the theoretical model and measure this phenomenon numerically.

The model we shall apply, which is one of many, is the Valenta-Sukiennicki model [9] considering the pairwise interactions between atoms. Based on our previous considerations [32, 34] we decide to use the extended Valenta–Sukiennicki model, which includes both the first and second neighbours' interactions. The aim of this paper is to compare the results of our calculations obtained earlier [31, 32], when the Lennard-Jones potential was considered with the new results when we apply the Morse potential instead.

In this study we shall consider AB_3 fcc alloys and ABC_2 bcc alloys. First we recall the ternary alloys description within the Valenta–Sukiennicki model [33]. Then, the binary alloys are seen as special cases of ternary alloys.

We also make a note on binary A_xB_{1-x} fcc alloys of arbitrary atoms concentrations.

Introduction

The study of thin films has been very intense during the last decades. One of the reasons is the increase of technical possibilities to investigate them experimentally. The other one is enormous technical applications.

It is observed, both theoretically and experimentally that in thin films samples the concentrations of atoms differ between the inner and outer layers in various temperatures [5, 7, 9, 18]. There are four classical surface effects worth considering: segregation, relaxation (change of distance between layers), reconstruction (change of atom ordering of the surface), and adsorbtion [17]. In this paper we shall inves-

tigate the segregation effects in chosen binary and ternary alloy thin films and we shall neglect the other three phenomena as they shall probably be of small influence on the segregation effect [17]. By the word "segregation" we shall mean the phenomenon of occurence of different concentrations of atoms in layers in a broad range of temperatures [47].

There are various theoretical approaches to thin films phenomena [7]. Below we present one of them, the Valenta–Sukiennicki model [9]. We shall consider the pairwise interactions between atoms including both the first and second neighbours. As the first neighbours interact more strongly than the second ones, we shall propose proportionality coefficients necessary in order to this fact into account: the first approach has been made before [32] with the use of the Lennard-Jones potential. In this paper we shall also include the Morse potential and compare the numerical results for the two approaches.

In our previous paper [32] in case of AB_3 fcc alloy and the Lennard-Jones potential included, small corrections were obtained when the second or the second and third nearest neighbours were included compared to the results obtained for the first neighbours only. However, if we include the 2nd neighbours in case of ABC_2 bcc alloy, the corrections, compared to the case with 1st neighbours only, are quite big. This phenomenon can be explained in the following way: it can be estimated, that in case of the first alloy the second neighbours are adding about 6% of sum of interactions of the first neighbours, while the third neighbours add about 7% more to both the first and second neighbours interactions. These numbers are not big. In the case of the second alloy the situation is different: we may estimate that the influence of 2nd neighbours is about 30% of the 1st ones [33]. The third neighbours seem to be adding about 7% more, so their interactions can be omitted. In numerical calculations for ABC_2 bcc alloys we have obtained big differences in the graphs of long-range order parameters and atoms concentrations in layers between the situations when the first neighbours were included, and the first and second neighbours were included.

Based on these considerations we decide to use the extended Valenta-Sukiennicki model including both the first and the second neighbours interactions. We shall consider two alloys: AB_3 fcc alloy and ABC_2 bcc alloy, both of (111) surface orientation. We pose the following question: assuming we consider the first and second neighbours, shall we obtain similar numerical results – in terms of atoms concentrations in layers and long-range order parameters if, in place of the Lennard-Jones potential included in earlier papers [32, 33] we include the Morse potential?

The first alloy considered, AB_3 fcc of (111) surface orientation, we divide into equal monoatomic layers (c.f. Fig. 1 (a)), the second one, ABC_2 bcc alloy of (111) surface orientation, we also divide into equal layers, each consisting of four consecutive monoatomic layers Fig. 1 (b)).

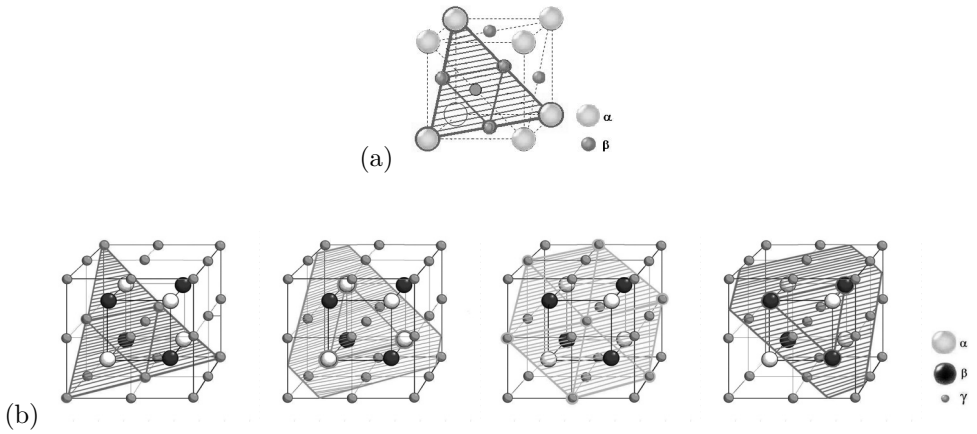


Fig. 1: The structure of one layer in the case of (a) AB_3 fcc alloy of (111) surface orientation; one layer is equal to one monoatomic layer, (b) in the case of ABC_2 bcc alloy of (111) surface orientation; one layer is composed of four consecutive monoatomic layers.

Valenta-Sukiennicki model for ternary alloys

Below we briefly recall the Valenta–Sukiennicki model for ternary alloys [33]. Given an infinite (in the plane) thin film of an arbitrary ternary alloy $A_xB_yC_z$, $x+y+z=1$, we divide the system into n identical, not necessarily monoatomic layers parallel to the surface. We assume each layer consists of N atoms. The whole lattice consists of three sublattices: α, β and γ, where the α (β or γ) lattice consists in stochiometric and ordered case of all A (B or C, respectively) atoms. The relative numbers of α (β, γ) sites are equal to F_α (F_β or F_γ respectively) and

$$F_\alpha + F_\beta + F_\gamma = 1.$$

Namely, we assume

$$F_\alpha = x = F_A, \quad F_\beta = y = F_B, \quad F_\gamma = z = F_C, \tag{1}$$

where F_A, F_B and F_C denote the relative numbers (concentrations) of A, B, C atoms in the alloy.

Let p_X^σ denote the probability that a site σ in i-th layer is occupied by an atom X, $\sigma \in \{\alpha, \beta, \gamma\}$, $X \in \{A, B, C\}$, $i = 1, 2, \ldots, n$. We have:

$$\begin{aligned} p_A^\alpha(i) + p_B^\alpha(i) + p_C^\alpha(i) &= 1, \\ p_A^\beta(i) + p_B^\beta(i) + p_C^\beta(i) &= 1, \\ p_A^\gamma(i) + p_B^\gamma(i) + p_C^\gamma(i) &= 1. \end{aligned} \tag{2}$$

The concentrations of A atoms (B or C atoms) in layers $i = 1, 2, \ldots, n$ are given by:

$$
\begin{aligned}
z_A(i) &= F_\alpha p_A^\alpha(i) + F_\beta p_A^\beta(i) + F_\gamma p_A^\gamma(i),\\
z_B(i) &= F_\alpha p_B^\alpha(i) + F_\beta p_B^\beta(i) + F_\gamma p_B^\gamma(i),\\
z_C(i) &= F_\alpha p_C^\alpha(i) + F_\beta p_C^\beta(i) + F_\gamma p_C^\gamma(i).
\end{aligned} \tag{3}
$$

The system has also to satisfy:

$$
\sum_{i=1}^{n} z_A(i) = nF_A, \quad \sum_{i=1}^{n} z_B(i) = nF_B, \quad \left(\sum_{i=1}^{n} z_C(i) = nF_C\right). \tag{4}
$$

The last equation follows from the other two.

We define the long-range order parameters in the following way:

$$
\begin{aligned}
t_A(i) &= \frac{p_A^\alpha(i) - z_A(i)}{1 - F_\alpha},\\
t_B(i) &= \frac{p_B^\beta(i) - z_B(i)}{1 - F_\beta},\\
t_C(i) &= \frac{p_C^\gamma(i) - z_C(i)}{1 - F_\gamma}
\end{aligned} \tag{5}
$$

for $i = 1, 2 \ldots, n$. In a completely ordered state we have $t_X(i) = 1$, whilst in a completely disordered state the probabilities of finding, say, an A atom in α, β or γ sites are the same, so $t_X(i) = 0$, for $X \in \{A, B, C\}$ and $i = 1, 2, \ldots, n$. The three long-range order parameters are linearly independent and their number cannot be reduced.

Through the diffusion process the atoms in the lattice tend to obtain positions which are better from the point of view of total energy of the system. The equilibrium values of $t_X(i)$, $z_X(i)$, $X \in \{A, B, C\}$, $i = 1, 2, \ldots, n$, are obtained when the free energy of the system given by

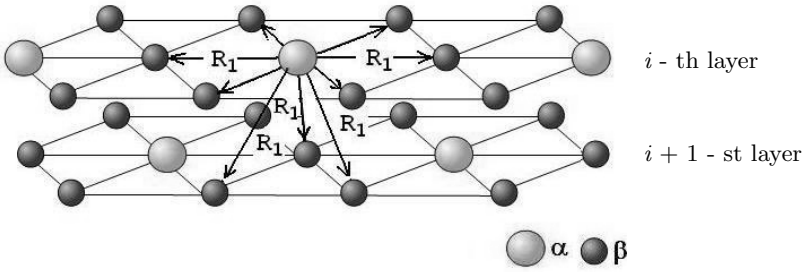
$$
F = U - TS \tag{6}
$$

is minimized. U denotes the internal energy, T absolute temperature, S entropy of the system.

Internal energy

In Bragg-Williams approximation the internal energy of the system is given as the average over the energies corresponding to a given long-range order. We need to know the mean number of pairs of nearest neighbours in given layer and neighbouring layers. We shall be considering only the pairwise interactions between atoms.

Let $-v_{XY}$ denote the interaction energy between atoms X and Y and let $\langle XY\rangle^{i,j}$ denote the number of pairs of neighbouring atoms X and Y such that X is in i-th layer, Y is in $i+j$-th layer. The internal energy of the thin film consisting of n layers is equal to:



$$r^0_{\alpha\alpha}(R_1) = 0, \;\; r^0_{\alpha\beta}(R_1) = 6, \;\; r^1_{\alpha\alpha}(R_1) = 0, \;\; r^1_{\alpha\beta}(R_1) = 3.$$

Fig. 2: Coefficients $r^0_{\alpha\alpha}(R_1)$, $r^0_{\alpha\beta}(R_1)$ and $r^1_{\alpha\alpha}(R_1)$, $r^1_{\alpha\beta}(R_1)$ in the lattice of AB_3 fcc alloy with orientation (111).

$$U = -\sum_{1\le i\le n}\left(\langle XX\rangle^{i,0} v_{XX} + \langle XY\rangle^{i,0} v_{XY}\right)$$

$$-\frac{1}{2}\sum_{i,j>0}\left(\langle XX\rangle^{i,+j} v_{XX} + \langle XY\rangle^{i,+j} v_{XY}\right) \tag{7}$$

$$-\frac{1}{2}\sum_{i,j>0}\left(\langle XX\rangle^{i,-j} v_{XX} + \langle XY\rangle^{i,-j} v_{XY}\right),$$

where

$$X, Y \in \{A, B, C\},\ X \neq Y,\ v_{XY} = v_{YX} \text{ and } i, i \pm j \in \{1, 2, \ldots, n\}.$$

Certainly, the interaction energy $-v_{XY}$ depends not only on the type of atoms but also on the distance between them. In order to include different pairs of atoms situated at different distances we shall make appropriate calculations based either on the Lennard-Jones or the Morse potentials.

Number of pairs of neighbours

Let us fix distance R between the sites of the alloy lattice. Let $r^j_{\sigma\tau}$ denote the number of neighbours of an atom in σ site in i-th layer which are in τ site in $i+j$-th layer and the distance between σ and τ sites equals R. The meaning of $r^j_{\sigma\tau}$ coefficients for the smallest distance, R_1, between the sites of the lattice of AB_3 fcc alloy of the orientation (111) is shown in Fig. 2. $R_1 = a\frac{\sqrt{2}}{2}$, where a is the lattice constant.

The number of pairs $\langle XY\rangle^{i,+j}$ of neighbours is given by the formula [33]:

$$
\begin{aligned}
\langle XY\rangle^{i,+j} &= \tfrac{1}{2}NF_\alpha\left(p_X^\alpha r_{\alpha\alpha}^j p_Y^{\alpha,+j} + p_X^\alpha r_{\alpha\beta}^j p_Y^{\beta,+j} + p_X^\alpha r_{\alpha\gamma}^j p_Y^{\gamma,+j}\right)\\
&+ \frac{1}{2}NF_\beta\left(p_X^\beta r_{\beta\alpha}^j p_Y^{\alpha,+j} + p_X^\beta r_{\beta\beta}^j p_Y^{\beta,+j} + p_X^\beta r_{\beta\gamma}^j p_Y^{\gamma,+j}\right)\\
&+ \tfrac{1}{2}NF_\gamma\left(p_X^\gamma r_{\gamma\alpha}^j p_Y^{\alpha,+j} + p_X^\gamma r_{\gamma\beta}^j p_Y^{\beta,+j} + p_X^\gamma r_{\gamma\gamma}^j p_Y^{\gamma,+j}\right),
\end{aligned}
\tag{8}
$$

where

$$
p_X^\sigma = p_X^\sigma(i), \quad p_Y^{\tau,+j} = p_Y^\tau(i+j), \quad X,Y \in \{A,B,C\},
$$
$$
\sigma,\tau \in \{\alpha,\beta,\gamma\}, \quad i,\, i+j = 1,2,\ldots,n, \quad j = 0,\pm1,\pm2.
$$

The values of the coefficients $r_{\sigma\tau}^j$ concerning the first and second neighbours for both the AB_3 fcc and ABC_2 bcc alloys of (111) orientations have been given in [32, 33].

Entropy

According to the well known Boltzmann formula, entropy is given by

$$
S = k_{\mathrm{B}} \ln g, \tag{9}
$$

where k_{B} denotes the Boltzmann constant and g denotes the number of possible configurations for given concentrations: $z_A(i)$, $z_B(i)$, $z_C(i)$, and long-range order parameters: $t_A(i), t_B(i), t_C(i)$, for $i = 1,2,\ldots,n$. Thus, for ternary alloy we have [33]

$$
g = \prod_{i=1}^{n} g_i, \quad \text{where}
$$

$$
g_i = \begin{pmatrix} NF_\alpha \\ NF_\alpha p_A^\alpha \end{pmatrix}\begin{pmatrix} NF_\alpha(1-p_A^\alpha) \\ NF_\alpha p_B^\alpha \end{pmatrix}\begin{pmatrix} NF_\beta \\ NF_\beta p_B^\beta \end{pmatrix}\begin{pmatrix} NF_\beta(1-p_B^\beta) \\ NF_\beta p_A^\beta \end{pmatrix}\cdot
$$
$$
\cdot\begin{pmatrix} NF_\gamma \\ NF_\gamma p_C^\gamma \end{pmatrix}\begin{pmatrix} NF_\gamma(1-p_C^\gamma) \\ NF_\gamma p_A^\gamma \end{pmatrix}.
$$

Further we obtain:

$$
S = k_{\mathrm{B}} \ln \prod_{i=1}^{n} \left\{ \frac{(NF_\alpha)!}{(NF_\alpha p_A^\alpha)!(NF_\alpha p_B^\alpha)!(NF_\alpha p_C^\alpha)!} \cdot \right.
$$
$$
\left. \cdot \frac{(NF_\beta)!}{(NF_\beta p_A^\beta)!(NF_\beta p_B^\beta)!(NF_\beta p_C^\beta)!} \cdot \frac{(NF_\gamma)!}{(NF_\gamma p_A^\beta)!(NF_\gamma p_B^\beta)!(NF_\gamma p_C^\beta)!} \right\}, \tag{10}
$$

where $p_X^\tau = p_X^\tau(i)$.

Binary alloy as a special case of ternary alloy

Let us consider a system of n layers of a binary alloy A_xB_{1-x} with an arbitrary surface orientation. The system can be seen as a special, degenerate case of a ternary alloy, if we 'remove' both C atoms and the corresponding γ sites, i.e. we assume

$$F_C = F_\gamma = 0, \;\; r^j_{\sigma\gamma} = r^j_{\gamma\sigma} = 0, \quad p^\gamma_X(i) = p^\sigma_C(i) = 0 \quad \text{for } \sigma \in \{\alpha, \beta, \gamma\}, \tag{11}$$

$i = 1, 2, \ldots, n,$, $j = 0, \pm 1, \pm 2$. By substituting (11) to the formulae (2), (3), (8), (10) for ternary alloys we obtain their binary alloys counterparts, all of which might as well be obtained independently, by straightforward calculations [9]. Thus, in binary alloys the system of equations describing probabilities (2) becomes

$$\begin{aligned} p^\alpha_A(i) + p^\alpha_B(i) &= 1, \\ p^\beta_A(i) + p^\beta_B(i) &= 1, \end{aligned} \tag{12}$$

the system of equations describing concentrations of atoms in layers (3) becomes

$$\begin{aligned} z_A(i) &= F_\alpha p^\alpha_A(i) + F_\beta p^\beta_A(i), \\ z_B(i) &= F_\alpha p^\alpha_B(i) + F_\beta p^\beta_B(i). \end{aligned} \tag{13}$$

The formula (8) for the number of pairs of neighbours is reduced to

$$\begin{aligned} \langle XY \rangle^{i,+j} = NF_\alpha \left(p^\alpha_X r^j_{\alpha\alpha} p^{\alpha,+j}_Y + p^\alpha_X r^j_{\alpha\beta} p^{\beta,+j}_Y \right) \\ + NF_\beta \left(p^\beta_X r^j_{\beta\alpha} p^{\alpha,+j}_Y + p^\beta_X r^j_{\beta\beta} p^{\beta,+j}_Y \right), \end{aligned} \tag{14}$$

where

$$\begin{gathered} p^\sigma_X = p^\sigma_X(i),\, p^{\tau,+j}_Y = p^\tau_Y(i+j), \;\; X, Y \in \{A, B\}, \;\; \sigma, \tau \in \{\alpha, \beta\}, \\ i = 1, 2, \ldots, n, \quad j = 0, \pm 1, \pm 2 \quad i,\, i + j \in \{1, 2, \ldots, n\}. \end{gathered}$$

We also have

$$t_A(i) = \frac{p^\alpha_A(i) - z_A(i)}{1 - F_\alpha} = p^\alpha_A(i) - p^\beta_A(i), \tag{15}$$

$$t_B(i) = \frac{p^\beta_B(i) - z_B(i)}{1 - F_\beta} = p^\beta_B(i) - p^\alpha_B(i) = p^\alpha_A(i) - p^\beta_A(i) = t_A(i), \tag{16}$$

which means that in every layer the system is now characterized by only one long-range order parameter. The formula (10) for entropy becomes

$$S = k_{\mathrm{B}} \ln \prod_{i=1}^{n} \frac{(NF_\alpha)!(NF_\beta)!}{(NF_\alpha p^\alpha_A)!(NF_\alpha p^\alpha_B)!(NF_\beta p^\beta_A)!NF_\beta p^\beta_B)!}. \tag{17}$$

where $p^\tau_X = p^\tau_X(i)$.

Including the first and second neighbours

Both in the case of ternary and binary alloys we are interested in including the interactions between the first and the second neighbours. Denote by R_1, R_2 the distance between the first and second neighbours, respectively. For a given distance R_j, $j = 1, 2$, determining the coefficients $r^j_{\sigma\tau}(R_j)$ is an easy geometric task. The geometrical structures of the first and second neighbours of every site in two cases:

(a) of binary AB_3 fcc alloy with orientation (111) and (b) of ABC_2 bcc alloy with orientation (111) are illustrated in Fig. 3 (a) and (b), respectively.

If we wish to include both the first and second neighbours then the general formula for internal energy should be rewritten as

$$\begin{aligned} U &= -\sum \left(\langle XY \rangle_{R_1} v_{XY}(R_1) + \langle XY \rangle_{R_2} v_{XY}(R_2) \right) \\ &= -\sum \left(\langle XY \rangle_{R_1} + \langle XY \rangle_{R_2} \frac{v_{XY}(R_2)}{v_{XY}(R_1)} \right) v_{XY}(R_1) \\ &= -\sum \langle XY \rangle v_{XY}(R_1) \end{aligned} \tag{18}$$

where $\langle XY \rangle_{R_1}$ and $\langle XY \rangle_{R_2}$ denote the pairs of atoms which are the first and second neighbours, respectively, $v_{XY}(R_1)$ and $v_{XY}(R_2)$ denote the interaction energy between the pair of atoms X and Y when they are situated at the distance of R_1 or R_2, respectively, and we also denote

$$\langle XY \rangle := \langle XY \rangle_{R_1} + \langle XY \rangle_{R_2} \frac{v_{XY}(R_2)}{v_{XY}(R_1)} \tag{19}$$

which includes both the number of first neighbours and a fraction of number of second neighbours in proportion to the (diminished) strength of a second neighbours pair interaction with relation to the first neighbours pair. In the light of the new formula (18) it is natural to define

$$\varepsilon := \frac{v_{XY}(R_2)}{v_{XY}(R_1)} \tag{20}$$

which will further be called the *relative interaction coefficient*. Let us note ε depends both on the distances R_1 and R_2 which are different in different types of alloys and atoms, and probably on the type of potential we include: either the Lennard-Jones or the Morse potential. The different values of ε will further be calculated.

Having the value of ε we can easily calculate the number of pairs of neighbours according to formulae (8) or (14) appropriately, by putting

$$r^j_{\sigma\tau} = r^j_{\sigma\tau}(R_1) + \varepsilon r^j_{\sigma\tau}(R_2). \tag{21}$$

In this way the first and second neighbours are taken together, while the influence of the second neighbours is diminished by the factor ε.

Calculating the relative interaction coefficients

Lennard-Jones potential. If we include the Lennard-Jones potential given by the formula

$$V(R) = \varepsilon \left(\left(\frac{\rho}{R} \right)^{12} - \left(\frac{\rho}{R} \right)^{6} \right), \tag{22}$$

where ρ denotes the distance at which the potential equals 0, then it seems reasonable to neglect the quickly diminishing repulsive forces and include only the attractive

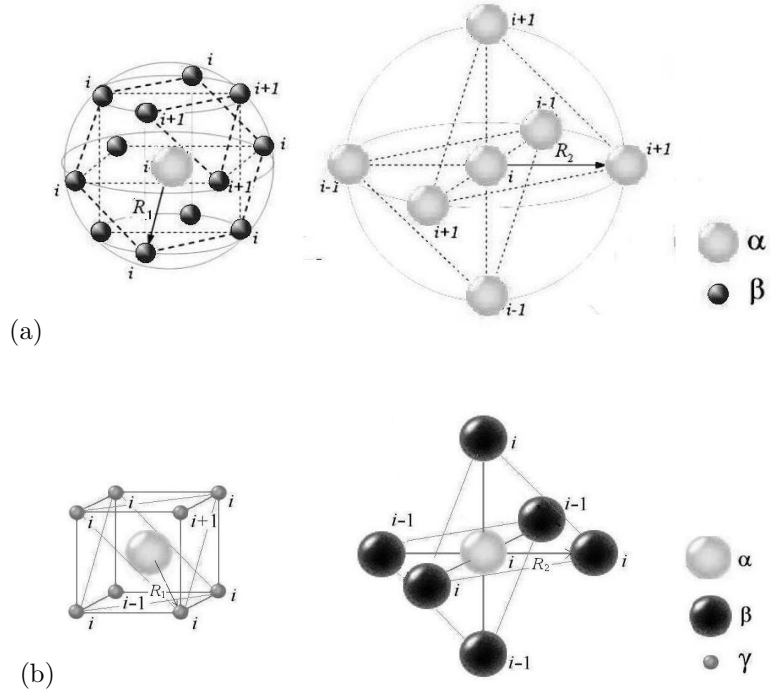


Fig. 3: Figure 3. The geometrical structure of the first and second neighbours of every site in two cases: (a) binary AB_3 fcc alloy with orientation (111), $R_1 = \frac{\sqrt{2}}{2}a$, $R_2 = a$, where a denotes the lattice constant; (b) ABC_2 bcc alloy with orientation (111), $R_1 = \frac{\sqrt{3}}{4}a$, $R_2 = \frac{1}{2}a$, where a denotes the lattice constant.

van der Waals forces, proportional to the inverse of the sixth power of the distance between atoms. Therefore the relative interaction coefficient is equal to

$$\varepsilon_{L-J} = \frac{v_{XY}(R_2)}{v_{XY}(R_1)} = \frac{(R_1)^6}{(R_2)^6}. \tag{23}$$

Consequently, in the case of AB_3 fcc alloy we have

$$\varepsilon_{L-J} = \frac{(\frac{\sqrt{2}}{2}a)^6}{(a)^6} = 0.125$$

and ABC_2 bcc alloy we have

$$\varepsilon_{L-J} = \frac{(\frac{\sqrt{3}}{4}a)^6}{(\frac{1}{2}a)^6} \approx 0.422\,.$$

Morse potential. We have the following formula for the Morse potential [24]:

$$\tilde{\Phi}_M(l) = C_M \left[\frac{1}{\alpha} x_1 \exp(\alpha y) - \frac{1}{\beta} x_2 \exp(\beta y)\right], \tag{24}$$

where

$$C_M = \frac{\alpha\beta D_0}{\alpha - \beta}, \quad x_1 = \exp[\alpha(1 - l/r_0)], \quad x_2 = \exp[\beta(1 - l/r_0)],$$

α, β are expressing the slope of the potential curve, r_0 and D_0 define the minimum of the potential and are respectively equal to the equilibrium distance of two isolated atoms and their dissociation energy. About the renormalizing parameter y, described in the paper of Malinowska-Adamska *et al* [24, 25] we assume it is equal to zero.

In the case of the Morse potential included, due to many necessary coefficients we cannot give an universal formula for the value of ε_M valid for all types of alloys and atoms. Instead, we shall make use of the paper of Malinowska-Adamska *et al* [45] in which the coefficients for some kinds of atoms have been calculated.

In case of AB_3 fcc alloys we substitute into the formula (24) the coefficients for Ni atoms given in [24]:

$$\alpha = 10.54, \quad \beta = 5.24, \quad r_0 = 2.549 \cdot 10^{-10} \text{ m}, \quad D_0 = 45.6 \cdot 10^{-21} \text{ J}$$

and we obtain the value of ε_M for fcc alloys

$$\varepsilon_M = 0.2199.$$

In the case of ABC_2 bcc alloys we similarly obtain

$$\varepsilon_M = 0.691.$$

Assumptions about the interactions between atoms

Binary alloys. Throughout the paper we consider the interactions between atoms A and B: v_{AA}, v_{BB} and $v_{AB} = v_{BA}$. We shall define the following V parameter:

$$V = v_{AB} - \frac{1}{2}(v_{AA} + v_{BB}). \tag{25}$$

V has positive value for ordering and negative value for segregating alloys. Additionally we define

$$\Delta = (v_{BB} - v_{AA})/V. \tag{26}$$

Like in [9] we shall always assume $V > 0$, $\Delta = 0$.

Ternary alloys. We shall assume the interaction energies between atoms satisfy

$$v_{XX} = v_{YY} \quad \text{and} \quad v_{XY} > v_{XX} \quad \text{if} \quad X \neq Y, \quad X, Y \in \{A, B, C\}. \tag{27}$$

We define the following V parameter

$$V = v_{AB} + 2v_{AC} + 2v_{BC} - (v_{AA} + v_{BB} + 2v_{CC})/2. \tag{28}$$

We shall make our calculations for the alloy which satisfies

$$v_{AB} = v_{AC} = v_{BC} = 1.25 \cdot v_{AA}. \tag{29}$$

Numerical results

In the case of AB_3 fcc the graphs of the long-range order parameters and atoms concentrations in layers obtained when the potential taken into account was either the Lennard-Jones or the Morse potential seem to be similar (c.f. Fig 4). Regardless of the type of potential included we observe the following phenomena: the alloy is perfectly ordered at the temperature of 0^oK; with the rise of temperature the differences of concentrations in layers appear; the A atoms tend to have bigger concentrations in the interior layers than in the exterior ones; The concentrations of atoms in layers 1 and 6, 2 and 5, 3 and 4 are identical; there exists temperature at which all the long-range order parameters drop to zero. The most visible difference is that the dropping temperature of the long-range order parameters is lower in the case of Morse potential than in the case of the Lennard-Jones potential included.

If we consider the ABC_2 bcc alloy of (111) surface orientation and the two potentials, then again we observe relative similarity. And, again, the long-range order parameters drop temperature is lower in the case of Morse potential than in the case of Lennard-Jones potential included (c.f. Fig 5).

A note on binary alloys with arbitrary atoms concentrations

In this section we would like to make corrections to the subject considered previously [34].

Let us consider binary A_xB_{1-x} fcc alloy with (111) surface orientation and an arbitrary atoms concentration, $F_A = x$. As it has been stated before, in this type of alloy the second neighbours do not modify the results remarkably, so let us consider only the pairwise interactions between atoms which are the first neighbours. Assuming that some, although unknown, kind of order in the temperature of 0^oK in the alloy exists [21], let us denote by α (or β) those sites of the lattice which, at the temperature of 0^oK are occupied by A atoms (or B atoms, respectively). We have

$$x = F_A = F_\alpha, \quad F_B = F_\beta = 1 - x \tag{30}$$

which means that the number of sites equals the number of atoms. The concentrations of atoms in layers, long-range order parameters, the number of pairs of nearest neighbours, and entropy can be calculated according to the corresponding formulae (11)-(17) for binary alloys. The only problem is that we cannot determine the coefficients $r^j_{\sigma\tau}$, $\sigma, \tau \in \{\alpha, \beta\}$ appearing in the formula (14). We shall, however, find their *average* value in the lattice.

The dependence between the coefficients $r^j_{\alpha\alpha}$, $r^j_{\alpha\beta}$, $r^j_{\beta\alpha}$, $r^j_{\beta\beta}$. We have

$$r^j_{\alpha\alpha} + r^j_{\alpha\beta} = r^j_{\beta\alpha} + r^j_{\beta\beta} = K^j, \tag{31}$$

where K^j is the lattice constant, here: $K^0 = 6$ (i.e. every site has 6 neighbours in the same layer), $K^1 = 3$ (i.e. every site has 3 neighbours in the next layer) (cf. Fig. 2).

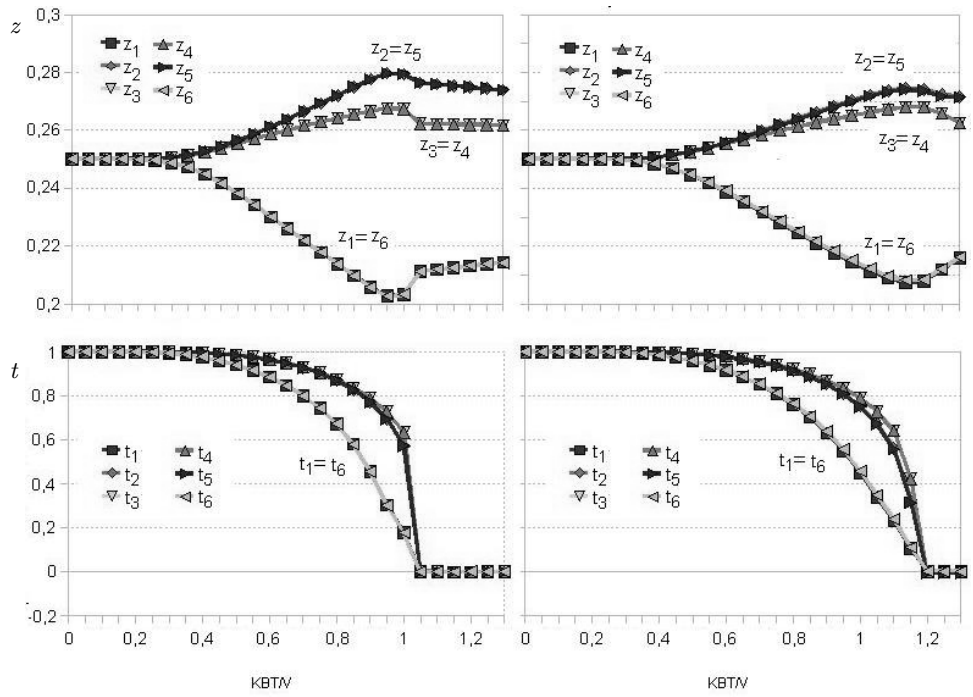


Fig. 4: AB_3 fcc alloy of (111) surface orientation, first and second neighbours included. Concentrations of A atoms in layers 1-6 and long-range order parameters in dependence of temperature. The graphs were obtained for two different potentials in question: the graphs on the left refer to the Morse potential, the graphs on the right refer to the Lennard-Jones potential.

Let us consider all the ordered pairs (α_k, σ_l), where α_k denotes an arbitrary (one of NF_α sites) α site in i-th layer, while σ_l denotes one of N sites in $i+j$-th layer (or $i-j$-th layer) and such that, the between α_k and σ_l is equal to the distance of nearest neighbours. The number of all the pairs (α_k, σ_l) in one layer is equal to $NF_\alpha K^j$. The same number can be obtained in a different way, namely, by considering that there are

$NF_\alpha r^j_{\alpha\alpha}$ of those pairs in which σ_l is an α site and

$NF_\beta r^j_{\beta\alpha}$ of those pairs in which σ_l is a β site.

Therefore we have

$$NF_\alpha K^j = NF_\alpha r^j_{\alpha\alpha} + NF_\beta r^j_{\beta\alpha}, \tag{32}$$

and hence

$$r^j_{\beta\alpha} = \frac{F_\alpha}{F_\beta}\left(K^j - r^j_{\alpha\alpha}\right) = \frac{x}{1-x}\left(K^j - r^j_{\alpha\alpha}\right). \tag{33}$$

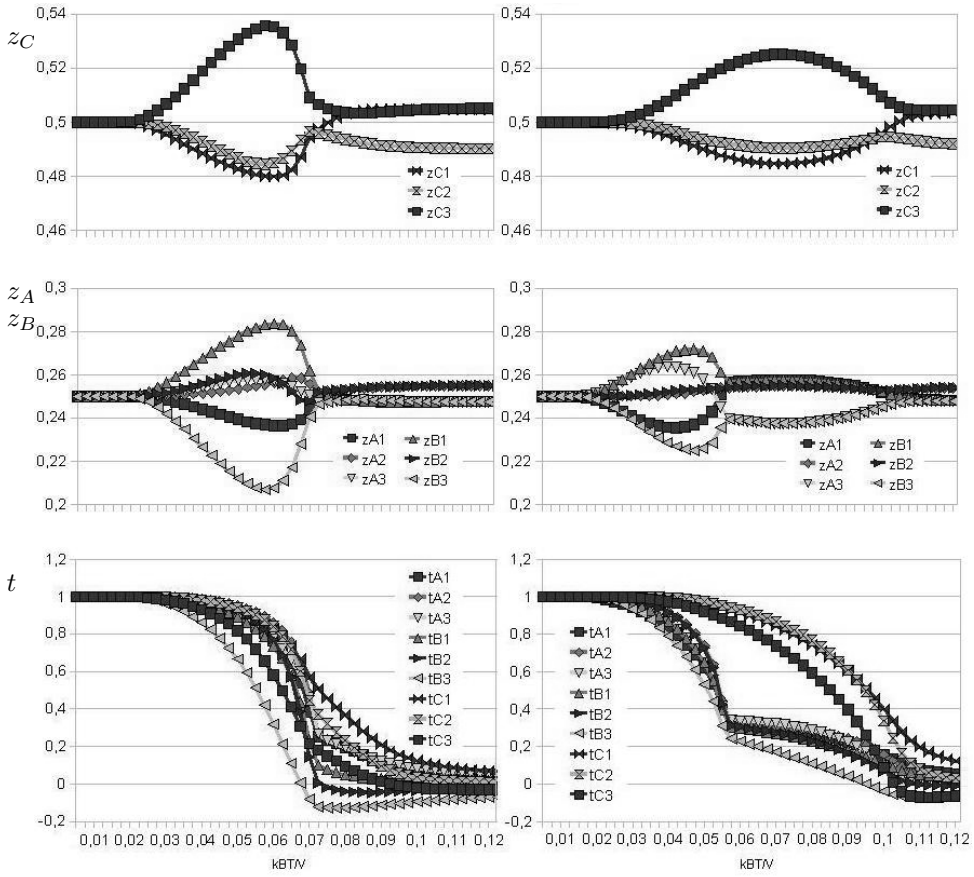


Fig. 5: ABC_2 bcc alloy of (111) surface orientation, first and second neighbours included. Concentrations of C atoms in layers 1-3 (top graphs), concentrations of A and B atoms in layers 1-3 (middle graphs) and long-range order parameters in dependence of temperature (lowest graphs). The graphs were obtained for two different potentials in question: the graphs on the left refer to the Morse potential, the graphs on the right refer to the Lennard-Jones potential.

Similarly, considering the number of all the pairs (β_k, σ_l) we obtain

$$NF_\beta K^j = NF_\alpha r^j_{\alpha\beta} + NF_\beta r^j_{\beta\beta}, \tag{34}$$

and hence

$$r^j_{\beta\beta} = K^j - \frac{F_\alpha}{F_\beta} r^j_{\alpha\beta} = K^j - \frac{x}{1-x} r^j_{\alpha\beta} = \frac{1-2x}{1-x} K^j + \frac{x}{1-x} r^j_{\alpha\alpha}. \tag{35}$$

From the above follows that in the alloy $A_x B_{1-x}$, if the value of $r^j_{\alpha\alpha}$ is known, then the other coefficients $r^j_{\sigma\tau}$ can be calculated from (31), (33), (35).

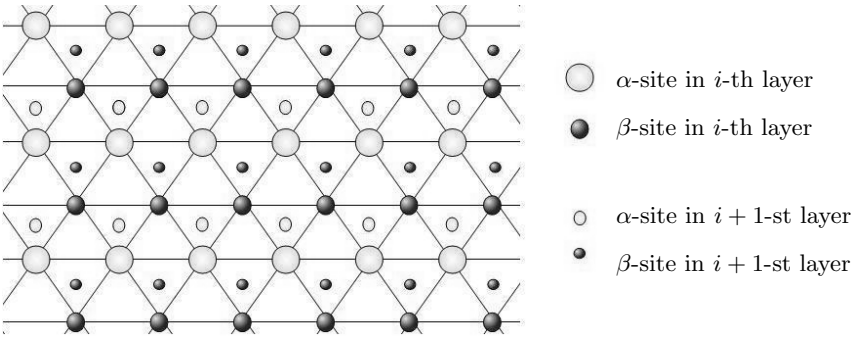


Fig. 6: One layer together with its neighbouring layer structure in the case of $A_{0.5}B_{0.5}$ fcc alloy with (111) surface orientation.

Case 1. $x < 0.25$. Since the atoms A and B have bigger interaction energy (according to (25)) the alloy has tendency to order, so the A atoms are more likely to spread in the sample than to aggregate in one place. As there are fewer A atoms now than in the alloy AB_3 fcc considered before, we can assume that, like in AB_3 fcc alloy, every α site is surrounded only by β sites, so we have

$$r^0_{\alpha\alpha} = 0, \quad r^1_{\alpha\alpha} = 0, \tag{36}$$

and the other coefficients follow from (31), (33), (35).

Case 2. $0.25 < x \leq 0.5$. As it can be seen in Fig. 3, in the case of AB_3 alloy all the first neighbours of an atom in α-site are in β-sites, but all its second neighbours are in α-sites. If the number of α sites grows bigger than 25%, which can be visualized as removing some β-sites and replacing them with α-sites, then all these 'new' α sites must become the first neighbours. So it is no longer possible to surround every α site with only β sites.

Let us consider a lattice consisting of 50% of α sites and 50% of β sites. We have the average of $r^0_{\alpha\alpha} = 2$, $r^1_{\alpha\alpha} = 1$ (sf. Fig. 6).

Denote by $r^j_{\alpha\alpha}(x)$ the mean value of $r^j_{\alpha\alpha}$ when the concentration of A atoms in the sample is equal to x, $0.25 < x \leq 0.5$. We have $r^0_{\alpha\alpha}(0.25) = 0$, $r^0_{\alpha\alpha}(0.5) = 2$. Assuming the dependence between the A atoms concentration in the sample, x, and the values of $r^0_{\alpha\alpha}(x)$ can be modeled by a linear function, we obtain that

$$r^0_{\alpha\alpha}(x) = 8x - 2. \tag{37}$$

Similarly, if we consider $r^1_{\alpha\alpha}(0.25) = 0$, $r^1_{\alpha\alpha}(0.5) = 1$, we obtain

$$r^1_{\alpha\alpha}(x) = 4x - 1. \tag{38}$$

The other coefficients follow from (31), (33), (35).

The values of $r^j_{\sigma\tau}$ coefficients calculated for different values of A atoms concentrations in the sample are given in Table 1.

x	$r^0_{\alpha\alpha}$	$r^0_{\alpha\beta}$	$r^0_{\beta\alpha}$	$r^0_{\beta\beta}$	$r^1_{\alpha\alpha}$	$r^1_{\alpha\beta}$	$r^1_{\beta\alpha}$	$r^1_{\beta\beta}$
0.05	0	6	0.32	5.68	0	3	0.16	2.84
0.1	0	6	0.67	5.33	0	3	0.33	2.67
0.15	0	6	1.06	4.94	0	3	0.53	2.47
0.2	0	6	1.5	4.5	0	3	0.75	2.25
0.25	0	6	2	4	0	3	1	2
0.3	0.4	5.6	2.4	3.6	0.2	2.8	1.2	1.8
0.4	1.2	4.8	3.2	2.8	0.6	2.4	1.6	1.4
0.5	2	4	4	2	1	2	2	1

Table 1. The values of $r^j_{\sigma\tau}$ coefficients for different values of A atoms concentrations in the A_xB_{1-x} alloy with (111) surface orientation.

The calculations have been made for the alloys A_xB_{1-x} with the following A atoms concentrations in the sample:

$$x = 5\%,\ 10\%,\ 15\%,\ 20\%,\ 25\%,\ 30\%,\ 40\%,\ 50\%.$$

The concentrations of A artoms in 5 layers and the corresponding long-range order parameters are given in Figs. 7 and 8.

It can be observed in all the graphs of long-range order parameters that their values suddenly drop to zero at certain temperature and it can be noticed that the smaller the A atoms concentration in the sample, the lower the temperature of the drop of the long range order parameters. Moreover, all the alloys, apart form the 50% alloy, show the effect of surface segregation which is realized by the diminished A atoms concentrations in the outer layers. In the 50% alloy there is no segregation effect, but the order transition can be observed, like in all other alloys.

In all the alloys considered in the whole range of temperatures we have

$$z_A(1) = z_A(5) \leq z_A(3) \leq z_A(2) = z_A(4), \tag{39}$$

and the equality is obtained at zero degrees temperature up till some other value of it.

The alloys of different A atoms concentrations differ also with respect to maximal difference in atoms concentrations in layers. The biggest difference can be observed in 5% alloy where we have

$$z_1(A) = z_5(A) = 0\%,$$

whereas in the second and fourth layers we have

$$z_2(A) = z_4(A) = 9.5\%.$$

With the rise A atoms concentration in the alloy, the maximal difference diminishes, up till vanishing completely in the alloy of 50% concentration of A atoms.

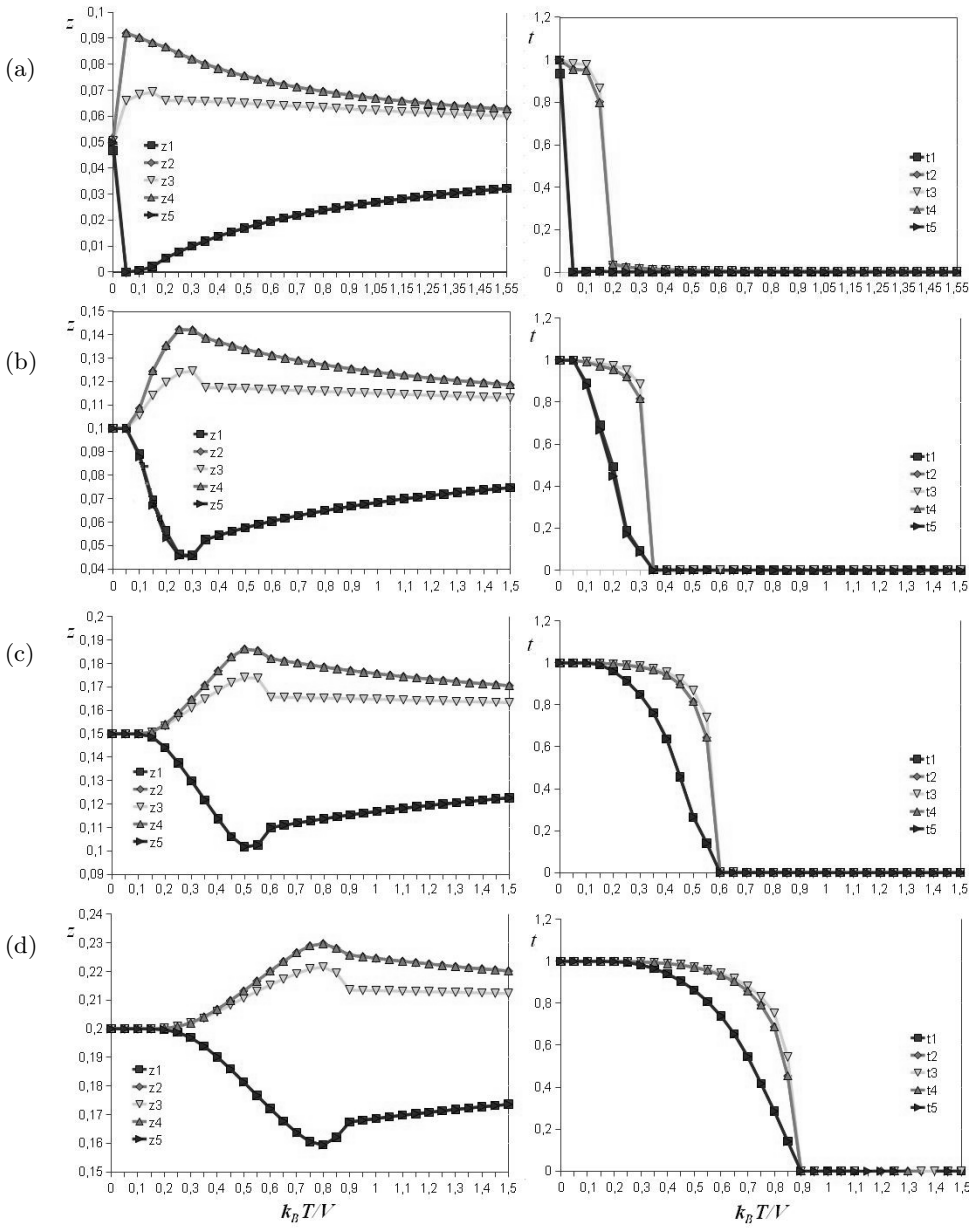


Fig. 7: Concentrations of A atoms in layers (graphs on the left) and long range order parameters (graphs on the right) in $A_x B_{1-x}$ fcc alloy with (111) surface orientation and with different A atoms concentrations, x, in the alloys: (a) $x = 0.05$, (b) $x = 0.1$, (c) $x = 0.15$, (d) $x = 0.20$.

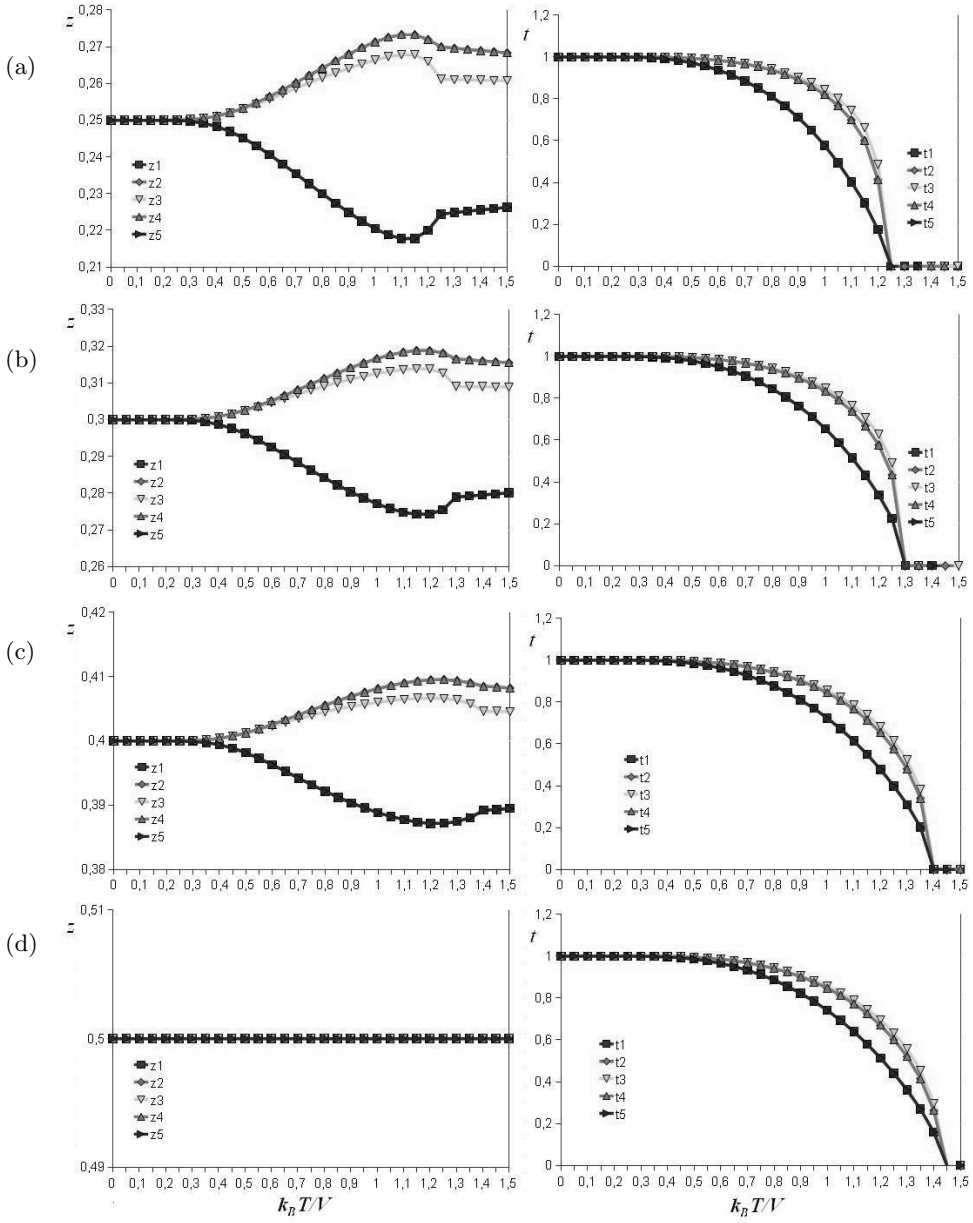


Fig. 8: Concentrations of A atoms in layers (graphs on the left) and long range order parameters (graphs on the right) in $A_x B_{1-x}$ fcc alloy with (111) surface orientation and with different A atoms concentrations, x, in the alloys: (a) $x = 0.25$, (b) $x = 0.3$, (c) $x = 0.4$, (d) $x = 0.5$.

References

[1] A. I. Anselm, *Osnovy statisticheskoi fiziki i termodinamiki*, Nauka, Moscow 1973.

[2] W. L. Bragg and E. J. Williams, *The effect of thermal agitation on atomic arrangement in alloys*, Proc. Roy. Soc. London, **145A** (1934), 699–730.

[3] W. M. Bartczak, S. Romanowski, M. Łandwijt, *Modelowanie oddziaływania wodoru z metalami przejściowymi za pomocą teorii funkcjonałów gęstości (DFT). Perspektywa kwantowych modeli katalizy heterogenicznej*, Wiadomości Chemiczne **55** (2001), 629–655.

[4] H. A. Bethe, *Statistical theory of superlattices*, Proc. Roy. Soc. London A **150** (1935), 552–575.

[5] A. Bienenstock , J. Lewis, *Order-Disorder of Nonstoichiometric Binary Alloys and Ising Antiferromagnets in Magnetic Fields*, Phys. Rev. **160** (1967), 393–403.

[6] L. Boltzmann, *Vorlesungen ber Gastheorie*, J.A. Barth, Leipzig, 1896 and 1898.

[7] M. Brejnak and P. Modrak, *Electronic theory of surface segregation for dilute transition metal alloys: predictions based on rigid-band-like approach*, Surf. Sci. **247** (1991) 215–221.

[8] J. R. Chelikowsky, Surf. Sci. **139** (1984), L197.

[9] F. L. Castillo Alvarado, A. Sukiennicki, L. Wojtczak, and Ilona Zasada, *Order–disorder phenomena in binary alloy thin films*, Physica B **344** (2004), 477–488.

[10] F. L. Castillo Alvarado, A. Kazibudzka, and J. Ławrynowicz, *Trade turnover and binary alloys as chaotic dynamical systems*, Bull. Soc. Sci. Lettres Łódź **52** Sér. Rech. Déform. **36** (2002), 67–84.

[11] J. E. Frąckowiak, *Badania nadstruktur typu* B2 *i* D03 *metodą efektu Mössbauera*, Wydawnictwo Uniwersytetu Śląskiego, Katowice 1993.

[12] J. Friedel, Adv. Phys. **3** (1954), 446.

[13] O. L. J. Gijzeman, Surf. Sci. **150** (1985), 1.

[14] A. I. Gusev, *Orderdisorder transformations and phase equilibria in strongly nonstoichiometric compounds,* Phys.-Usp. **43** (2000) 1–37.

[15] R. S. Ingarden, A. Jamiołkowski, and R. Mrugała, *Statistical Physics and Thermodynamics* (in Polish), PWN, Warszawa 1990.

[16] R. S. Ingarden, J. Ławrynowicz, *Physical and biophysical interpretations of a new model of open systems with superluminal velocities*, Acta Physicae Superficierum **6** (2004), 91–106.

[17] A. Kiejna and A. F. Wojciechowski, *Metal Surface Electron Physics*, Pergamon, Oxford 1996.

[18] J. C. Hamilton *Prediction of surface segregation in binary alloys using bulk alloy variables*, Phys. Rev. Lett. **42** (1979) 989992.

[19] R. Kikuchi, *A theory of cooperative phenomena*, Phys. Rev. **81** (1951), 988–1003.

[20] C. Kittel, *Introduction to Solid State Physics*, 3rd ed., Wiley, New York 1966.

[21] E. V. Kozlov, D. M. Shtern, N. M. Kormin, and O. A. Gudin, *Ordering of nonstoichiometric solid solutions*, J. Struc. Chem. **18** (1977), 64–69.

[22] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Pergamon Press Ltd, 1951.

[23] A. Maidou and H. M. Polatoglou, *Monte Carlo study of the* [001] *surface of* Cu_3Au *for* $T \neq 0$ *K*, Phys. Rev. B **60** (1999), 9145–9156.

[24] C. Malinowska-Adamska, *Self-consistent calculations of the dynamic and thermodynamic properties of rare-gas solids*, Acta Phys. Hung. **59** (1986), 257-272.

[25] C. Malinowska-Adamska, P. Słoma, and J. Tomaszewski: J. Phys. Condensed Matter **18** (2006) 751.

[26] C. Malinowska-Adamska and P. Słoma, *Self-consistent calculations of the dynamic, thermodynamic and elastic properties of a lattice of atomic crystals*, Sci. Bull. Tech. Univ. Łódź, 1010 Physics, **28** (2007), 33–44.

[27] S. Mamica and H. Puszkarski, *Nearest and next-nearest neighbourhood surface mapping in thin cubic films. Properties of structural sums,* Acta Phys. Superficierum **9** (2006), 17–70.

[28] J. L. Morán-López and K. H. Bennemann, *Surface effects on the order-disorder phase transition of A_3B alloys*, Phys. Rev. B **15** (1977), 4769–4780.

[29] J. L. Morán-López and L. M. Falicov, *Theory of surface effects in binary alloys. I. Ordering alloys*, Phys. Rev. B **18** (1978), 2542–2548.

[30] P. M. Morse, *Diatomic molecules according to the wave mechanics*, II. *Vibrational levels*, Phys. Rev. **34** (1929), 57-64.

[31] M. Nowak-Kępczyk, *Binary alloy thin films. The concepts of entropy and nearest neighbours description*, Bull. Soc. Sci. Lettres Łódź **55** Sér. Rech. Déform. **48** (2005), 141–155.

[32] M. Nowak-Kępczyk, *On entropy and internal energy in alloy thin films*, Acta Physicae Superficierum **10** (2008), 25–44.

[33] M. Nowak-Kępczyk, *Binary and ternary alloy alloy thin films. Entropy and internal energy in dependence on surface orientation and nearest neighbours approximation*, in preparation.

[34] M. Nowak-Kępczyk, *Binary alloy thin films with arbitrary atom concentrations and vacancies*, Bull. Soc. Sci. Lettres Łódź **59** Sér. Rech. Déform. **58** (2009), 87–99.

[35] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical recipes in C. The Art of Scientific Computing*, 2nd ed. Cambridge Univ. Press, Cambridge 1992.

[36] A. Rabenau (Ed.), Contributors: J. S. Anderson [and others], *Problems of Nonstoichiometry*, Amsterdam, North-Holland Publ. Co. 1970.

[37] S. Romanowski, W. M. Bartczak, and R. Wesołkowski, *Density functional calculations of the hydrogen adsorption on transition metals and their alloys. An Application to Catalysis*, Langmuir **15** (18) (1999), 57735780.

[38] S. Romanowski, T. Pietrzak and W. M. Bartczak, *Molecular dynamics calculations for a series of transition metals and their alloys*, Bull. Polish Acad. Sci. Chemistry **46**, no. 4 (1998), 397–410.

[39] S. J. Romanowski and J. F Gomes, *Monte-Carlo studies of the chemisorption and work function-temperature effects on noble-metals,* Journal of Electroanalytical Chemistry **1-2** (1994), 133–140.

[40] A. V. Ruban and H. L. Skriver, *Calculated surface segregation in transition metal alloys*, Computational Materials Science **15** (1999), 119–143.

[41] E. Schrödinger, *Statistical Thermodynamics*, 2nd ed., Cambridge Univ. Press, Cambridge 1952; repr. 1957 and 1960.

[42] T. Siklós, Acta Phys. Acad. Sci. Hung. **30** (1971), 181.

[43] J. Sokolov, F. Jona, and P. M. Marcus, *Multilayer relaxation of a clean* bcc Fe111 *surface*, Phys. Rev. B **33** (1986), 1397–1400.

[44] A. Sukiennicki, L. Wojtczak, Ilona Zasada, F. L. Castillo Alavarado, *Ferromagnetic thin films of binary alloys*, J. of Magnetism and Magnetic Materials **288** (2005), 137–145.

[45] P. Słoma, C. Malinowska-Adamska, J. Tomaszewski, *Thermodynamic properties of solid Neon in the reduced all-neighbours approximation of the self-consistent phonon theory*, Scientific Bulletin. Physics / Technical University of Łódź, Vol. **28** (2007), 69–77.

[46] C. Tsallis, M. Gell-Mann, and Y. Sato, *Extensivity and entropy production*, Europhysicsnews, vol. 36, **6** (2005), 186–189.

[47] U. Vahalia, P. A. Dowben, and A. Miller, *Surface segregation in binary alloys*, J. Vac. Sci. Technol. A, Vol. 4, No. **3** (1986), 1675–1679.

[48] L. Valenta and A. Sukiennicki, *Order-disorder phenomena at the surface of solids*, Phys. Stat. Sol. **17** (1966), 903–909.

[49] L. Wojtczak, Ilona Zasada, A. Sukiennicki, and F. L. Castillo Alavarado, *Order-disorder phenomena at the surfaces of alloy thin films*, Bull. Soc. Sci. Lettres Łódź **51** Sér. Rech. Déform. **35** (2001), 41–49.

High School of Business
Kolejowa 22, PL-26-600 Radom
Poland

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on July 16, 2010

CIENKIE WARSTWY STOPU PODWÓJNEGO A POTENCJAŁY LENNARDA-JONESA I MORSE'A

NOTATKA O STOPACH PODWÓJNYCH O DOWOLNEJ KONCENTRACJI ATOMÓW

S t r e s z c z e n i e

Rozważane są cienkie warstwy stopu podwójnego AB_3 o orientacji (111) oraz stopu potrójnego ABC_2 bcc o orientacji (111). W pracy posługuję się modelem Valenty-Sukiennickiego dla stopów potrójnych. W tym modelu stop podwójny może być traktowany jako szczególny (zdegenerowany) przypadek stopu potrójnego. Wartości koncentracji atomów w warstwach oraz wartości parametrów dalekiego zasięgu są wyznaczane numerycznie. Według obliczeń, zarówno koncentracje atomów w warstwach jak i wartości parametrów dalekiego zasięgu dla warstw wewnętrznych i zewnętrznych różnią się w szerokim zakresie temperatur.

Celem pracy jest porównać rezultaty numeryczne otrzymane, gdy we wzorze na energię swobodną układu uwzględniamy potencjał Morse'a z rezultatami otrzymanymi dla potencjału Lennarda-Jonesa. W drugiej części pracy wykonuję obliczenia koncentracji atomów w warstwach oraz parametrów dalekiego zasięgu w przypadku stopów A_xB_{1-x} fcc o dowolnej koncentracji atomów A w próbce.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2010 Vol. LX

Recherches sur les déformations no. 2

pp. 93–115

Roman Stanisław Ingarden and Julian Ławrynowicz

FINSLER-GEOMETRICAL MODEL OF QUANTUM ELECTRODYNAMICS II

PHYSICAL INTERPRETATION OF SOLENOIDAL AND NONSOLENOIDAL CONNECTIONS ON THE CANONICAL PRINCIPAL FIBRE BUNDLES

Summary

In the first part of the paper we have analysed different physical demands showing the importance of nonriemannian geometries in quantum electrodynamics. In this part, after outlining the five-dimensional Kałuża-Klein-like model, we study ferroelectric crystals in a Finsler geometry including the sine-Gordon equation for the surface. Then, within the generalized Yang-Mills equations we distinguish and discuss their solenidal and nonsolenoidal parts including, in the spirit of equipping the Finsler geometry with suitable nonlinear connections, the connections corresponding to the Yang-Mills field vs. Yang-Mills field itself. Next we consider the case of an arbitrary symmetry within the group $\mathrm{SO}(r+1,s)$ and the problem of simplifying the external field in terms of the metric and connection. We finish with an example and an outlook of future perspectives.

10. A concept of the five-dimensional model of nonlinear electrodynamics

In Sec. 1 of [IŁ4] we have already announced Beil's results [Be1, 2] about a relationship between the Finsler and Kałuża-Klein gauge theories. In a flat space without the scalar field it is natural to consider the Kałuża-Klein metric

$$\left(g_{\hat{\alpha}\hat{\beta}}\right) = \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_{nu} & 1 \end{pmatrix}$$

with $\mu, \nu = 1, 2, 3, 0$. Kerner [K] had proposed the full Lagrangian

$$\mathcal{L} = R + \gamma \left(R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} R^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} - 4 R_{\hat{\alpha}\hat{\beta}} R^{\hat{\alpha}\hat{\beta}} + R^2 \right),$$

including the Gauss-Bonnet term already mentioned in Sec. 1, with Y being a parameter characterizing the strength of nonlinearity. When expressed in four dimensions in terms of the Maxwell tensor, it becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{3}{16}\gamma\left[(F_{\mu\nu}F^{\mu\nu})^2 - 2F_{\mu\lambda}F_{\nu\rho}F^{\mu\nu}F^{\lambda\rho}\right].$$

The equations of motion are

$$F_{\lambda\rho|\mu} + F_{\rho\mu|\lambda} + F_{\mu\lambda|\rho}, \tag{61}$$

which correspond to the Bianchi identities.

The Lagrangian written in terms of the familiar fields $\mathbf{E}$ and $\mathbf{B}$ reads [KBG]:

$$\mathcal{L} = \frac{1}{2}\left(\mathbf{B}^2 - \mathbf{E}^2\right) + \frac{3}{2}\gamma\left(\mathbf{E}\cdot\mathbf{B}\right)^2$$

and the equations (61) may be rewritten as

$$\mathrm{div}\mathbf{E} = -3\gamma\mathbf{B}\cdot\mathrm{grad}(\mathbf{E}\cdot\mathbf{B}), \quad \mathrm{rot}\mathbf{B} = \partial_0\mathbf{E} + 3\gamma\left[\mathbf{B}\partial_0(\mathbf{E}\cdot\mathbf{B}) - \mathbf{E}\times\mathrm{grad}(\mathbf{E}\cdot\mathbf{B})\right].$$

In this approach, equivalent (cf. Sect. 1) to a simple Finsler-geometrical approach, the density of charge and the current are created by non-linearity of the field:

$$\rho = -3\gamma\mathbf{B}\cdot\mathrm{grad}(\mathbf{E}\cdot B), \quad \mathbf{j} = 3\gamma\left[\mathbf{B}\partial_0(\mathbf{E}\cdot B) - \mathbf{E}\times\mathrm{grad}(\mathbf{E}\cdot B)\right].$$

The continuity equation reads $\partial_0\rho + \mathrm{div}\,\mathbf{j} = 0$ and the Poynting vector conserves its usual form $\mathbf{S} = \mathbf{E}\times\mathbf{B}$. Yet, the energy density is modified:

$$\mathcal{E} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) + \frac{3}{2}\gamma\left(\mathbf{E}\cdot\mathbf{B}\right)^2$$

and the corresponding continuity equation reads $\partial_0\mathcal{E} + \mathrm{div}\,\mathbf{S} = 0$. One wave propagates in a Maxwellian way; the other possible wave solution is delayed.

11. Ferroelectric crystals in a Finsler geometry

In the case of a chain of atoms in a crystal in the Hamiltonian derived we may distinguish the terms $H_{\mathbf{e}}$, $H_{\mathbf{p}}$, and $H_{\mathbf{ep}}$ due to electronic vibrations, phonon vibrations, and electron-phonon coupling, respectively:

$$H_{\mathbf{e}} = \sum_{\lambda,\tilde{\lambda}} t_{\lambda,\tilde{\lambda}} C^+_\lambda C_\lambda + \frac{1}{2}U\sigma C_{\lambda(\sigma)}C_{\lambda(\sigma)}C^+_{\lambda(-\sigma)}C_{\lambda(-\sigma)} - \mu\sum C^+_\lambda C_\lambda,$$

$$H_{\mathbf{p}} = \frac{1}{2}\mathbf{M}\sum_j p_j^2 + \frac{1}{2}a\sum_j (R_{j+1}-R_j)^2 + \ldots, \quad H_{\mathbf{ep}} = I\sum_j (R_j - R_{j+1})\,(C^+_\lambda C_{\lambda+1}),$$

where

$$\lambda = (\nu, j, m, \sigma), \quad \lambda + 1 = (\nu, j+1, m, \sigma).$$

The expressions for $H_{\mathbf{e}}, H_{\mathbf{p}}$, and $H_{\mathbf{ep}}$ are written in the Hubbard approximation applied together with the pseudoharmonic approximation [GŁW2].

The equations of motion read

$$\dot{P}_j = -(\partial/\partial R_j)\langle H\rangle, \quad \dot{R}_j = (\partial/\partial P_j)\langle H\rangle,$$

where $H = H_{\mathbf{e}} + H_{\mathbf{p}} + H_{\mathbf{ep}}$ and we take the averaged value $\langle H\rangle$ because of the quantum statistical approach. Hence $\mathbf{M}\ddot{R}_j = -(\partial/\partial R)\langle H\rangle$ and, consequently,

$$\mathbf{M}\ddot{R}_j = \alpha(R_{j+1} + R_{j-1} - 2R_j) + \ldots * \ldots + I\sum_{\alpha}\Phi^*_{j\alpha}(\Phi_{j+1,\sigma} - \Phi_{j-1,\sigma}), \tag{62}$$

where $\ldots * \ldots$ denotes the terms related to the inhomogeneity of the system and the last term expresses the coupling of the crystallographic lattice with the system of electron. As in the quoted paper, the external force originates from the inhomogeneity of the phonon system, I and α are parameters, and σ stands for the fluctuation distribution.

For describing the motion of electrons we rearrange the Dirac-Maxwell system to an averaged Heisenberg-type equation:

$$i(d/dt)\Phi C_\lambda = [C_\lambda, H]$$

with

$$\Phi_\lambda = \langle 0|C_\lambda|0\rangle, \quad \Phi^*_\lambda = \langle 0|C^+_\lambda|0\rangle,$$

and hence

$$\begin{aligned} i(d/dt)\Phi_\lambda = {}& \mu\Phi_\lambda|\Phi_{\lambda(-\sigma)}|^2 - \mu\Phi_\lambda + t\,[\Phi_{\lambda+1} + \Phi_{\lambda-1}] \\ & + I(R_j - R_{j+1})\Phi_{\lambda+1} + I(R_{j-1} - R_j)\Phi_{\lambda-1}. \end{aligned} \tag{63}$$

When applying the continualized model of variables (the Lindner-Fedyanin method) the equations (62) and (63) imply

$$R_j/a \to x(\xi,t), \quad R_{j\pm1}/a \to x + x' + \frac{1}{2}x'',$$

so that

$$\mathrm{M}\ddot{x} = -\alpha x'' + 2\frac{I}{a}\sum_{\sigma}\frac{\partial}{\partial\xi}(\Phi_\sigma\Phi_{-\sigma}), \tag{64}$$

with $x = x(\xi,t)$ and

$$i\Phi_\sigma = T\Phi''_{-\sigma} + 2T\Phi_{-\sigma} - \frac{(2I)^2}{M(\omega^2 - \omega_0^2)}\sum_{\sigma'}\Phi^*_{\sigma'}\Phi_{-\sigma} + U|\Phi_{-\sigma}|^2\Phi_\sigma - \mu\Phi_\sigma, \tag{65}$$

where $\omega_0^2 = \alpha/\mathrm{M}$.

At least the third and fourth terms on the right-hand side of (65) cause the necessity of discussing the *solitary waves* in the context of the investigation of layers near to the surface in anharmonic crystals. The appearance of an electronic soliton is determined by the sign of the influence of vibrations according to the sign of the coefficients U and $-(2I)^2/\mathrm{M}(\omega^2 - \omega_0)^2$. By (64), the solution for vibrations $x = x(\xi,t)$ depends on the shape of solutions for Φ_σ and $\Phi_{-\sigma}$. We shall study the soliton solutions of the equations (64) and (65) with the force (as in (64)). The soliton equations have to be treated as equations of motion, i.e. as the Lagrange-Euler equations in spite of the fact that the Lagragian concerned does not exist in a finite-dimensional space even for the Korteweg-de Vries equations.

Fortunately we have built the Finsler-geometrical model of quantum electrodynamics, so that all the solitons are solutions of the equations of motion, i.e. of the Lagrange-Euler equations:

$$\frac{\partial}{\partial x}\left(u_t \frac{\partial}{\partial u_x}\mathcal{L}\right) = 0 \quad \text{with} \quad \mathcal{L} = \mathcal{L}(u, u_t, u_x), \quad u = u(x,t)$$

(equation (4.2) in [Ł5]). Then, in the case of ferroelectric crystals the results may be compared with those of Pouget and Maugin [PM1, 2]. Our solutions will express surface solitons if the maximum is attained on the surface of the crystal, i.e. at the end of the chain of atoms (in the sense of construction given in [GŁW2], pp. 48–49).

Let us concentrate on the case of single solitons and one-wall motion in elastic ferroelectrics in the presence of mechanical couplings. Then the equations (64) and (65) describing the solitary waves become

$$Qu_{tt} = \hat{c}_{\|} u_{xx}, \quad Qv_{tt} - c_{\perp} v_{xx} = -e\Theta_x \cos 2\Theta,$$
$$J\Theta_{tt} - K\Theta_{xx} = ev_x \cos 2\Theta + \chi \sin 2\Theta;$$

cf. [PM1, 2], where the physical meaning of the parameter functions is given. The quoted authors had shown, on the basis of continualized equations of the magnetoelasticity for ferroelectric crystals, that Bloch walls in an infinite crystal and Néel walls in a thin elastic film can be represented by "magnetoelastic" solitary waves.

In the first case the solitary waves are solutions of a simple sine-Gordon equation (cf. [Ł5], Sect. 4)

$$u_{xt} = \sin u. \tag{66}$$

In the second case the magnetic-spin orientation remains nonlinearly coupled to the elastic displacement polarized in the plane of the film. Therefore we study a nonlinearly coupled system of a sine-Gordon equation. It is still better to study a double sine-Gordon equation and two-wave equation. For such a system, Pouget and Maugin had obtained the solitary-wave solutions in closed form. As it has been announced, an important role in the staff is played by the *double sine-Gordon equation*

$$u_{xt} = \sin u - \gamma \sin 2u, \tag{67}$$

where γ is a real constant. If $\gamma = 0$, (67) reduces to (66). Equation (67) has one-soliton solutions

$$u_1 = -2\arctan[a\sinh(x - \omega t)], \quad u_2 = \pi - \arctan[a\sinh(x - \omega t)],$$

where ω is a constant and

$$a = 1/(1 + 2\gamma). \tag{68}$$

It is natural to ask: For which Lagrangians $\mathcal{L}$ in the sense of Finsler geometry, the Dirac-Maxwell equations of motion are satisfied by u_1 or/and u_2? The answer is given by [KLŁ]:

Proposition 1. *For*

$$L_1(u,u_x,u_t) = u_t u_x \left(1+\tan^2\frac{1}{2}u\right)^2 \Big/ \left(a^2+\tan^2\frac{1}{2}u\right),$$
$$L_2(u,u_x,u_t) = u_t u_x \left(1+\cot^2\frac{1}{2}u\right)^2 \Big/ \left(a^2+\cot^2\frac{1}{2}u\right),$$

as Lagrangians in the sense of Finsler geometry, the corresponding Dirac-Maxwell equations of motion read

$$2u_{tx} + u_t u_x \left(\tan\frac{1}{2}u\right)\left(2a^2-1+\tan^2\frac{1}{2}u\right) \Big/ \left(a^2+\tan^2\frac{1}{2}u\right) = 0 \tag{69}$$

and

$$2u_{tx} - u_t u_x \left(\cot\frac{1}{2}u\right)\left(2a^2-1+\cot^2\frac{1}{2}u\right) \Big/ \left(a^2+\cot^2\frac{1}{2}u\right) = 0, \tag{70}$$

respectively, while the functions u_1 and u_2 are special solutions of (69) *and* (70)*, respectively.*

We can see that for $\gamma = 0$, that is, by (68), for $a = 1$, the equations (69) and (70) become

$$u_t u_x = -2u_{tx}\cot\frac{1}{2}u \tag{71}$$

and

$$u_t u_x = 2u_{tx}\tan\frac{1}{2}u, \tag{72}$$

respectively, so indeed the equation (72) is a counterpart of the sine-Gordon equation in the Finsler geometry. Thus, thanks to the Finsler Lagrangian formalism, the generalized Pouget-Maugin description of solitary waves in ferroelectric crystals has directly been linked with basic symmetries of electron and nucleon systems. As shown in [KLŁ], by the Hurwitz pair theory (cf. [Ł3], Sect. B2), the description can be formulated in terms of Clifford hypercomplex analysis, and this is confirmed by the results of R. G. Beil, already quoted in Sect. 1 of [IŁ4] and in Sect. 10.

12. The Finsler-geometrical counterpart of the sine-Gordon equation for the surface

For a surface problem we shall use Hurwitz pairs in the simplest, euclidean case [Ł5], Sect. 5.2. Following [KLŁ] we are going to prove

Theorem 3. *Let (V,S) be the two-dimensional (Euclidean) Hurwitz pair. Let further $\mathcal{F} = \{f_1,\dots,f_N\}$ be a finite family of regular functions*

$$f_k : D \subseteq S \to V, \quad k = 1,\dots,N,$$

defined with respect to the above Hurwitz pair: $f_k = {f_k}^1 e_1 + {f_k}^2 e_2$, *where* (e_1, e_2) *is a basis of* V. *Then there exists a complex* $(N+1)\times(N+1)$*-matrix* $P_{\mathcal{F}}$ *such that*

$$P_{\mathcal{F}}^2 = P_{\mathcal{F}}, \quad P_{\mathcal{F}}^+ = P_{\mathcal{F}} \tag{73}$$

and

$$[P_{\mathcal{F}}, \Delta P_{\mathcal{F}}] = 0 \quad \textit{with} \quad \Delta = D\bar{D} = (\partial_1)^2 + (\partial_2)^2. \tag{74}$$

Explicitly, if $\Phi = (\varphi_1, \dots, \varphi_N)^T$ *is the vector in* $\mathbb{C}^N$ *of holomorphic functions (in the usual sense)* $\varphi_k = {f_k}^1 + i{f_k}^2$, $\quad k = 1, \dots, N$, *then*

$$P_{\mathcal{F}} = \frac{1}{1+\Phi^+\Phi}\begin{pmatrix} 1 & \Phi^+ \\ \Phi & \Phi\Phi^+ \end{pmatrix}. \tag{75}$$

Proof. To each family $\mathcal{F} = \{f_1, \dots, f_N\}$ of regular functions we can associate a vector $\Phi = (\varphi_1, \dots, \varphi_N)^T$, where $\varphi_k = {f_k}^1 + i{f_k}^2$, $k+1, \dots, N$, are holomorphic functions. This follows from the fact ([ŁR1], Theorem 3) that $Df_k = 0$ if and only if the vector $\Psi_k({f_k}^1, {f_k}^2)^T$ satisfies the equation

$$\left(I_2\partial^1 - i\sigma_2\partial^2\right)\Psi_k = 0, \tag{76}$$

where I_2 is the unit 2×2-matrix and σ_2 is the familiar second Pauli spin matrix; [Ł3], Sect. A1. Equation (76) is just equivalent to the system

$$\partial^1 {f_k}^1 = \partial^2 {f_k}^2, \quad \partial^2 {f_k}^1 = -\partial^1 {f_k}^2.$$

Consider now the $(N+1)\times(N+1)$-matrix $P_{\mathcal{F}}$ given by (75). It is a projector or the complex vector $(1, \Phi)^T$. Since Din and Zakrzewski [DZ] checked that $P_{\mathcal{F}}$ satisfies (74), the proof is completed.

Remark 1. There exists a hyperbolic counterpart of Theorem 3: [KLŁ], p. 36.

Let us turn our attention to the equation (72). The known solution of the sine-Gordon equation: the simple soliton solution and the double soliton solutions are also solutions of (72). For a satisfactory description of the surface solitary waves we need solutions of (72) which do not satisfy (66). Indeed, we have:

Proposition 2. *The sine-Gordon equation* (66) *and its counterpart* (72) *in the Finsler geometry are not equivalent. However, all known solutions of* (66)*: the simple soliton solution and the double solutions are also solutions of* (72).

Proof. we shall verify that the function

$$v = 4\arctan\left[\exp(t + ax + b)\right],$$

for suitably chosen constants a and b, is a solution of (72), but does not satisfy (66). In fact, we check that

$$v_t = 4\exp\left(t + ax + b\right)/\left[1 + \exp 2(t + ax + b)\right]),$$
$$v_x = 4a\exp\left(t + ax + b\right)/\left[1 + \exp 2(t + ax + b)\right].$$

Hence the left-hand side of (72) equals

$$L = u_t u_x = 16u \exp 2\,(t + ax + b) \,/\, [1 + \exp 2(t + ax + b)]^2 .$$

On the other hand, we calculate:

$$v_{xt} = 4a \exp\,(t + ax + b)\,[1 - \exp 2(t + ax + b)] \,/\, [1 + \exp 2(t + ax + b)]^2 .$$

Hence the right-hand side of (72) becomes

$$\begin{aligned} R &= 2u_{xt} \tan \frac{1}{2} u \\ &= 2 \cdot 4a \exp(t + ax + b) \frac{1 - \exp 2(t + ax + b)}{[1 + \exp 2(t + ax + b)]^2} \cdot \frac{2 \exp(t + ax + b)}{[1 - \exp 2(t + ax + b)]} \\ &= 16a \exp 2(t + ax + b) \,/\, [1 + \exp 2(t + ax + b)]^2 . \end{aligned}$$

Hence $L = R$ and v is a solution of (72) for any a and b. If v were a solution of (66), it would have to satisfy it at every point (x, t). At $(0, 0)$ we have $v(0, 0) = 4 \arctan(\exp b)$. Setting $b = \ln \tan \frac{1}{8}\pi$, we get

$$v(0,0) = 4 \arctan\left(\tan \frac{1}{8}\pi\right) = \frac{1}{2}\pi, \quad \sin v(0,0) = \sin \frac{1}{2}\pi = 1. \tag{77}$$

Yet, the left-hand side of (66) is equal to

$$v_{xt}(0,0) = 4a \left(\tan \frac{1}{8}\pi\right) \left(1 - \tan^2 \frac{1}{8}\pi\right) \left(1 + \tan^2 \frac{1}{8}\pi\right)^2$$

and it can be quite arbitrary. For example, if we set

$$a = \frac{1}{2} \left(1 + \tan^2 \frac{1}{8}\pi\right)^2 \Big/ \left[\left(\tan \frac{1}{8}\pi\right) \left(1 - \tan^2 \frac{1}{8}\pi\right)\right],$$

then we get $v_{xt}(0, 0) = 2$. We have arrived to a contradiction with (77), as desired.

The proof of the second statement of the proposition consists in straightforward verification.

13. The solenoidal and nonsolenoidal parts of the generalized Yang-Mills equations as observed on the canonical principal fibre bundle

When transforming the Yang-Mills equations to a Finsler geometry it seems important, from the physical point of view, following the classical analogies, to distinguish the "solenoidal" and "nonsolenoidal" parts of those equations. Lemma 2 in [IŁ4], especially the decomposition appearing in (35), suggests the relationship of the parts distinguished with $\mathrm{Div} F)^j$ and $[A_k, F^{jk}]$, respectively ($j = 1, 2, \ldots, n$); Lemma 3 in [IŁ4], especially the decomposition appearing in (43), suggests the relationship of the parts distinguished with δF and $2\mathrm{Tr}_g(A \otimes_{\mathcal{G}} F)$, respectively. However, if we look for a suitable **global** decomposition of the final equation [IŁ4] (48), the proper choice is $d(\star_{\mathcal{G}} \Omega^2)$ for the "solenoidal" part and $\star_{\mathcal{G}} \Omega^2 \wedge_{\mathcal{G}} \omega$ for the "nonsolenoidal" part,

where – as before – $\wedge_{\mathcal{G}}$ denotes the $\mathcal{G}$-dependent wedge product operator. Indeed, we have the following corollary to Theorem 2 in [IŁ4] ([GKŁ]):

Corollary 1. *Under the hypotheses of Theorem 1, without assuming that* **A** *corresponds to a stationary value of* [IŁ4] (47)*, the decomposition*

$$D(\star_{\mathcal{G}}\Omega^2) = d(\star_{\mathcal{G}}\Omega^2) + \star_{\mathcal{G}}\Omega^2 \wedge_{\mathcal{G}} \omega, \tag{78}$$

where the both addends are well-defined global tensorial forms. An analogous statement for the decomposition [IŁ4] (53) *is, in general, false.*

Proof. We consider the global tensorial form $D(\star_{\mathcal{G}}\Omega^2)$. Without assuming that **A** corresponds to a stationary value of [IŁ4] (47), we can see by Theorem 2 or, more exactly, by the relations [IŁ4] (53) and [IŁ4] (55), that the form in question is equivalent to the left-hand side of [IŁ4] (43) with F and A being locally expressed by [IŁ4] (56), where (e_r) and (e_j^*) are as in the proof of Theorem 2. The forms F and A themselves have their values in the lie algebra $\mathcal{G}$ defined over co-ordinate neighbourhood U of $M_\#$ such that $(\pi^{-1}(U), \pi, U)$ is a trivial bundle ($M_\#$ is as in [IŁ4], Theorem 2).

The first statement of the corollary is evident. When proving the second statement, by [IŁ4] (55) we need to verify if vanishing of the one-form $\mathrm{Tr}_g(A \otimes_{\mathcal{G}} F)$ depends of the choice of local trivialization. Hence, consider another local trivialization over U. The forms F and A are then transformed according to the formulae

$$\tilde{F} = \hat{g}^{-1} F \hat{g} \quad \text{and} \quad \tilde{A} = \hat{g}^{-1} A \hat{g} + \hat{g}^{-1} d\hat{g},$$

respectively, where $\hat{g} : U \to \mathrm{GL}(\mathrm{m}, \mathbb{R})$ or $\mathrm{GL}(\mathrm{m}, \mathbb{C})$. Consequently,

$$\begin{aligned}
\mathrm{Tr}_g(\tilde{A} \otimes_{\mathcal{G}} \tilde{F}) &= \mathrm{Tr}_g \left[\left(\hat{g}^{-1} A \hat{g} + \hat{g}^{-1} d\hat{g} \right) \otimes_{\mathcal{G}} \left(\hat{g}^{-1} F \hat{g} \right) \right] \\
&= \mathrm{Tr}_g \left(\hat{g}^{-1} A \hat{g} \otimes_{\mathcal{G}} \hat{g}^{-1} F \hat{g} \right) + \mathrm{Tr}_g \left(\hat{g}^{-1} d\hat{g} \otimes_{\mathcal{G}} \hat{g}^{-1} F \hat{g} \right) \\
&= \mathrm{Tr}_g \left[\hat{g}^{-1} (A \otimes_{\mathcal{G}} F) \hat{g} \right] + \mathrm{Tr}_g \left\{ \hat{g}^{-1} \left[(d\hat{g}) \hat{g}^{-1} \otimes_{\mathcal{G}} F \right] \hat{g} \right\} \\
&= \hat{g}^{-1} \left[\mathrm{Tr}_g (A \otimes_{\mathcal{G}} F) \right] \hat{g} - \hat{g}^{-1} \left[\mathrm{Tr}_g (\hat{g} d\hat{g}^{-1} \otimes_{\mathcal{G}} F) \right] \hat{g}.
\end{aligned}$$

We conclude that the equation $\mathrm{Tr}_g(A \otimes_{\mathcal{G}} F) = 0$ is not invariant with respect to local trivializations and this completes the proof.

By Theorem 2 in [IŁ4], the global formulation of the generalized Yang-Mills problem, depending on an arbitrary metric g and an arbitrary non-abelian Lie group $\mathcal{G}$, involves in a natural way the bundle $P(M_\#, G)$ of orthonormal frames of **E**, which is equipped with the connection N_0 induced by a given G-connection $N_\#$ on **E**, corresponding to the $\mathcal{G}$-vector field **A**. As noted in [IŁ4], Sect. 1, especially when commenting the scheme (2) of the first named author, manipulations with connections are of basic importance for the proposed approach to quantum electrodynamics.

We can go still further and consider a more general situation with $P(M_\#, G)$ being an arbitrary principal fibre bundle of **E** – a real or complex G-vector bundle over $M_\#$, where G is an arbitrary compact subgroup of some $\mathrm{SO}(\mathrm{m})$ or $\mathrm{SU}(\mathrm{m})$.

Then we can construct on $\mathbf{E}$ the canonical riemannian [hermitian] metric h (cf. e.g. [W], p. 69) and consider the h-depending Hodge $*$-operator $\star_{\mathcal{G}}$. Finally, passing to a Finsler (in particular, Randers) metric can be done, in analogy to several previously given examples, by regarding the metric h a sort of *Fisher's information* or the (partial) metric of the *associated Riemannian space* (cf. e.g. [L7], pp. 121 and 123, especially Theorem 2, and papers quoted there). The distinction between the "solenoidal" and "nonsolenoidal" ports of the generalized Yang-Mills equation now motivates, even purely mathematically, the following

Definition. A connection $N_{\#}$ corresponding to a $\mathcal{G}$-vector field is called *solenoidal* if the corresponding connection matrix ω satisfies the condition

$$\star_{\mathcal{G}}\Omega^2 \wedge_{\mathcal{G}} \omega = 0. \tag{79}$$

Following [GKaŁ] we give another motivation to our definition:

Lemma 4. *Let φ be a horizontal tensorial two-form on the principal fibre bundle $P(M_{\#}, G)$, where $M_{\#}$ is the base space and G is an arbitrary compact subgroup of some* SO(m) *or* SU(m). *Then, for arbitrary fields X, Y, Z on P, we have*

$$\begin{aligned} D\varphi(X,Y,Z) = d\varphi(X,Y,Z) - \frac{1}{3}\{&[\varphi(X,Y),\omega(Z)] \\ &+ [\varphi(Y,Z),\omega(X)] + [\varphi(Z,X),\omega(Y)]\}. \end{aligned} \tag{80}$$

Proof. Obviously, every vector of P is the sum of the horizontal component and the vertical component. The both sides of (80) are bilinear and skew-symmetric in X, Y, Z, so it is enough to verify the equality in the following special cases:
(i) X, Y, Z are vertical,
(ii) X, Y, Z are horizontal,
(iii) X and Y are vertical, and Z is horizontal,
(iv) X and Y are horizontal, and Z is vertical.
By the definition of covariant differentiation and an obvious relation

$$\begin{aligned} d\varphi(X,Y,Z) &= \frac{1}{3}[X\varphi(Y,Z) + Y\varphi(Z,X) + Z\varphi(X,Y)] \\ &= [\varphi([X,Y],Z) + \varphi([Y,Z],X) + \varphi([Z,X],Y)] \end{aligned} \tag{81}$$

the proof in the cases (i)–(iii) is immediate.

In the remaining case (iv), without any loss of generality, we may assume that X and Y are horizontal lifts and $Z = A^*$ is the fundamental vector field corresponding to some A and G, where $\mathcal{G}$ is the Lie algebra corresponding to G. In consequence the left-hand side of (81) is zero, whereas the right-hand side equals

$$\frac{1}{3}A^*\varphi(X,Y) - \varphi([Y,A^*],X) - \varphi([A^*,X],Y) - \frac{1}{3}[\varphi(X,Y),\omega(A^*)].$$

Yet, X and Y are R_a-invariant as horizontal lifts with

$$R_a^*\varphi = ada^{-1}\varphi \quad \text{for} \quad a \in G. \tag{82}$$

Consider the one-parameter subgroup $\{G_t\}$ of G generated by A. Hence

$$[X, A^*] = \lim_{t\to 0^+} \frac{1}{t} [R_{a_t}(X) - X] \tag{83}$$

and thus

$$\begin{aligned} A^*_u \varphi(X,Y) &= \lim_{t\to 0^+} \frac{1}{t} [\varphi_{ua_t}(X,Y) - \varphi_u(X,Y)] \\ &= \lim_{t\to 0^+} \frac{1}{t} \left[(R^*_{a_t}\varphi)u(X,Y) - \varphi_u(X,Y)\right] \\ &= \lim_{t\to 0^+} \frac{1}{t} \left[\mathrm{ad}a_t^{-1}\varphi_u(X,Y) - \varphi_u(X,Y)\right] \\ &= -[A, \varphi_u(X,Y)]\,. \end{aligned}$$

Hence, in our case (81) yields

$$0 = \frac{1}{3} A^* \varphi(X,Y) - \frac{1}{3} [\varphi(X,Y), \omega(A^*)]\,,$$

so, by (82) and the R_a-invariance of X and Y, we conclude that

$$\varphi([Y, A^*], X) = 0 \quad \text{and} \quad \varphi([A^*, X], Y) = 0.$$

The lemma is proved.

Theorem 4. *A connection $N_\#$ corresponding to a $\mathcal{G}$-vector field, where $\mathcal{G}$ is the Lie algebra of a semi-simple Lie group G, is solenoidal if and only if*

$$\Omega^2 \equiv D\omega = 0. \tag{84}$$

If, in particular, the principal fibre bundle $P(M_\#, G)$, where $M_\#$ is the base space, is not trivial and admits a solenoidal connection, then $M_\#$ is **multiply connected**.

Proof. The sufficiency is evident. Since ω is solenoidal, then, by (49), $[\star_{\mathcal{G}}\Omega(X,Y), A] = 0$ for every A of G and horizontal fields X and Y on P. Yet, G is semi-simple, and this means that $\star_{\mathcal{G}}\Omega(X,Y) = 0$ for all horizontal fields X and Y on P. Hence, by the definition of the Hodge $*$-operator $\star_{\mathcal{G}}$ we obtain (84), as desired.

In addition to another motivation to the definition of a solenoidal $\mathcal{G}$-vector field, Theorem 4 motivates in an elegant way the corresponding assumption of multiple connectivity made in earlier papers by the second named author and L. Wojtczak [ŁW1, 2, Ł1], as well as by several other authors like Misner and Wheeler [MiW], Dirac [D], and Sakharov [Sa]. The interpretation of Theorem 4 confirms also physical conclusions presented in the quoted papers. In particular, by the above results the assumption about the multiple connectivity of the base space $M_\#$, treated as the space of the particle, is not arbitrary, but results in a rigorous way from the assumption of non-triviality of the principal fibre bundle $P(M_\#, G)$ [GH].

Furthermore, the possibility of obtaining the Yang-Mills equations, reduced to a form equivalent to the Maxwell equations appearing in some class of metrics, shows that the variety of physical fields can be treated as a result of geometry while their

sources are of the same nature by physical principles [Hu]. Theorem 4 elucidates also the problem due to the second named author [Ł2] of replacing a given Yang-Mills field, in general with currents, by a $\mathcal{G}$-vector field in a different curved manifold, by way of including current in the geometry, so that the field becomes Maxwell-like. This way of thinking perfectly agrees with the scheme [IŁ4] (2) of modifying properly a Finsler, in particular Randers metric and some nonlinear connection. For instructive example and comments we also refer to [ŁKW, KŁW].

14. Connections corresponding to the Yang-Mills field vs. the Yang-Mills field itself

Following our observation after [Ł6] in Sect. 13, we concentrate now on a p-dimensional compact orientable Riemannian manifold M_e with metric g^e and on a p-dimensional compact orientable pseudoriemannian manifold M_h with metric g^h. For the scheme [IŁ2] (2) of the first of us it is important to concentrate on a sort of Fisher's information: g^e or g^h, or on the associated Riemannian space M_e or pseudoriemannian space M_h. For $k = e$ and $k = h$, let $\mathbf{E}_k = (E_k, \pi_k, M_k)$ be a real or complex G_k-vector bundle over the base space M_k, where E_k denotes the bundle space, $\pi_k : E_k \to M_k$ is the projection and G_k an arbitrary subgroup of $\mathrm{SO}(r_k+1, s_k)$ with $r_k+1+s_k = p$, SO(m), or SU(m). Further, suppose that

$$p = 8\ell + 4, \quad \text{where} \quad \ell = 0, 1, 2, \dots .$$

In analogy to [ŁR2], Sect. 5, we can introduce the field of tetrads: If

$$z_k = (z_k^a) \quad \text{and} \quad \zeta_k(z_k) = (\zeta_k^a)(z_k)$$

then the *field of tetrads* λ_k can locally be expressed by the relations

$$\lambda_k = \sum_\alpha (\partial/\partial\zeta_k^\alpha)\, \lambda_{ka}^\alpha .$$

The (pseudo)riemannian tensor of $\mathbf{E}_k$, i.e. the (pseudo)riemannian metric of M_k, is locally given by

$$g_{ab}^k = \sum_\alpha \sum_\beta \lambda_{ka}^\alpha \lambda_{kb}^\beta \eta_{\alpha\beta}^k$$

where η^k is the metric of the tangent space. Now, consider the pair $(\mathbf{E}_k, V_k)$ such that at every point z of M_k its tangent space at z forms, together with V_k, a (psudo)Euclidean Hurwitz pair; cf. [Ł5], Sect. 5.2. Then $(\mathbf{E}_k, V_k)$ is called a (*pseudo*) *riemannian Hurwitz pair*; cf. [AŁR]. The concept can still be generalized by replacing V_k with a (psudo)riemannian or symplectic manifold whose tangent bundle $\{V_k\}$ consists of the spaces understood as before.

Let us turn, in particular, our attention to the case $G_k = \mathrm{SU}(2)$ with $\mathcal{G}^k = \mathrm{su}(2)$ denoting the corresponding Lie algebra. We extend the definition of a *Yang-Mills field* in a natural way: it is any su(2)-valued vector field $\mathrm{A} = (A_\alpha),\ \alpha = 1, \dots, p$, in an open set U in $\mathbb{R}^p$ [AŁ]. Take the bundle

$$P_k = P_k(M_k, G_k)$$

of orthonormal frames of M_k, endowed with the connection N_0^k induced by a given connection $N_\#^k$ on M_k, corresponding to a $\mathcal{G}^k$-vector field A_k on M_k. Denote by ω_k and Ω_k^2 the connection matrix and curvature form corresponding to A_k, respectively. Let D_k stand for the covariant derivative operator related to N_0^k, $\star g^k$ for the g^k-dependent Hadge $*$-operator, and

$$\star_{\mathcal{G}^k} : A^q P^k \to A^{p-q} P_k,$$

where $A^r P_k$ is the modulus of horizontal r-forms on P_k, of the type $\operatorname{ad} G_k$. We have [ŁKS]:

Theorem 5. *Consider a duality* (cf. [IŁ1], Sect. 5) *of the Yang-Mills generation of Hurwitz pairs, i.e. take into account one of the following configurations listed in Table* 1 *and in Table* 2*, where* $\rightleftharpoons$ *denotes the duality in question, and* κ *stands for the matrix determining the scalar product in* V_k*:*

$$(f_1, f_2)_{V_k} = f_1 \kappa f_2^T.$$

In order to explain the special values of κ *in the tables, set*

$$C_\alpha = i\gamma_\alpha C_t, \quad t \text{ fixed}, \quad t \in \{1, \dots, p\}, \quad \alpha \neq t;$$

$$C_\alpha = \left[c_{j\alpha}^\ell\right], \quad \sum_\alpha \sum_j c_{j\alpha}^\ell a^\alpha f_k^j = (a \circ f_k)^\ell$$

$$\text{for} \quad \ell = 1, \dots, \ \dim_{\mathbb{R}} V_k; \ k = e \text{ and } k = h,$$

where $\circ$ *denotes the multiplication in the Hurwitz pair* (E_k, V_k)*, consider the sequence of matrices*

$$\tilde{\gamma}_\alpha = \gamma_\alpha, \ \alpha = 1, \dots, r; \quad \hat{\gamma}_\beta = \gamma_{r+\beta}, \quad \beta = 1, \dots, s,$$

and, further, the real matrices

$$A = (-i)^r \tilde{\gamma}_1 \tilde{\gamma}_2 \dots \tilde{\gamma}_r, \quad B = (-i)^s \hat{\gamma}_1 \hat{\gamma}_2 \dots \hat{\gamma}_s.$$

If $r = -1, 0$*, we set* $A = I_n$ *(the* $n \times n$*-unit matrix), if* $s = 0$*, we set* $B = I_n$*. Hereafter* $\ell = 0, 1, 2, \dots$

Tab. 1. Configurations of the Yang-Mills generation

$r_e + 1 = 8\ell + 4$	$s_e = 0$	$\kappa_e = A$	$\rightleftharpoons$	$r_k + 1 = 1$	$s_k = 8\ell + 3$	$\kappa_k = iA\gamma_1$
$r_e + 1 = 8\ell + 4$	$s_e = 0$	$\kappa_e = B$	$\rightleftharpoons$	$r_k + 1 = 1$	$s_k = 8\ell + 3$	$\kappa_k = iB\gamma_1$
$r_e + 1 = 0$	$s_e = 8\ell + 4$	$\kappa_e = A$	$\rightleftharpoons$	$r_k + 1 = B\ell + 3$	$s_k = 1$	$\kappa_k = iA\gamma_2$
$r_e + 1 = 0$	$s_e = 8\ell + 4$	$\kappa_e = B$	$\rightleftharpoons$	$r_k + 1 = 8\ell + 3$	$s_k = 1$	$\kappa_k = iB\gamma_2$

Tab. 2. Configurations of the Yang-Mills generation (another presentation)

$$\begin{array}{ccccc}
(8\ell+4,\ 0;\ A) & \rightleftharpoons & (1,\ 8\ell+3;\ iA\gamma_1) & & \\
 & & \Updownarrow & & \\
 & & 1,\ 8\ell+3;\ iB\gamma_1) & \rightleftharpoons & (8\ell+4,\ 0;\ B) \\
 & & & & \\
 & & 8\ell+3,\ 1;\ iA\gamma_2) & \rightleftharpoons & (0,\ 8\ell+4;\ A) \\
 & & \Updownarrow & & \\
(0,\ 8\ell+4;\ B) & \leftrightharpoons & (8\ell+3,\ 1;\ iB\gamma_2) & &
\end{array}$$

For $k = e$ and $k = h$, let M_k be a G_k-vector bundle as specified before. Then

$$D_k\left(\star_{\mathcal{G}^k}\Omega_k^2\right) = d_k\left(\star_{\mathcal{G}^k}\Omega_k^2\right) + \star_{\mathcal{G}^k}\Omega_k^2 \wedge_{\mathcal{G}^k} \omega^k, \tag{85}$$

where the both addends are well-defined global tensorial forms with $\wedge_{\mathcal{G}^k}$ being the $\mathcal{G}^k$-dependent wedge product operator. Moreover, a connection $N^k_{\#}$ corresponding to A_k is solenoidal, that is the second addend in (85) *equals zero, if and only if*

$$\Omega_k^2 \equiv D_k\omega_k = 0.$$

If, in particular, the principal fibre bundle $P_k(M_k, G_k)$, where M_k is the base space, is not trivial and admits a solenoidal connection, then M_k is multiply connected.

Proof. Consider a global tensorial form $D_k(\star_{\mathcal{G}^k}\Omega_k^2$. In local frame, if x^a is a co-ordinate system in a co-ordinate neighbourhood of M_k, let

$$F = d_kA_k + A_k \wedge_{\mathcal{G}^k} A_k \quad \text{with} \quad A_k = \sum_\alpha A^k_\alpha dx^\alpha_k, \quad (A^k_\alpha) = A_k.$$

Since $N^k_{\#}$ is a G_k-connection corresponding to a $\mathcal{G}^k$-vector field A_k of class C^∞ on M_k, then

$$\delta_{g^k}F_k + 2\mathrm{Tr}_{g^k}\left(A_k \otimes_{\mathcal{G}^k} F_k\right) = 0, \quad \text{where} \quad \delta_{g^k} = *_{g^k}d*_{g^k},$$

and $\otimes_{\mathcal{G}^k}$ denotes the $\mathcal{G}^k$ -dependent tensor product operator; here

$$F_k = F^k_{\alpha\beta}d_kx^\alpha_k \wedge_{\mathcal{G}^k} d_kx^\beta_k \quad \text{with} \quad F^k_{\alpha\beta} = \partial^k_\alpha A^k_\beta - \partial^k_\beta A^k_\alpha + [A^k_\alpha, A^k_\beta].$$

Yet, we have

$$*_{g^k}\sigma_k^*D_k\left(\star_{\mathcal{G}^k}\Omega_k^2\right) = \delta_{g^k}F_k + *_{g^k}\left(A_k \wedge_{\mathcal{G}^k} *_{g^k}F^k\right) = 2\mathrm{Tr}\left(A_k \otimes_{\mathcal{G}^k} F_k\right),$$

where σ_k is a local cross-section of $P_k(M_k, G_k)$: $\sigma_k^*\Omega_k^2 = F_k$ and $\sigma_k^*\omega_k = A_k$; cf. [BLS] and [IŁ4], formulae (54) and (55). Consequently,

$$D_k\left(\star_{\mathcal{G}^k}\Omega_k^2\right) = \delta_{g^k}F_k + 2\mathrm{Tr}_{g^k}\left(A_k \otimes_{\mathcal{G}^k} F_k\right),$$

so we have proved (85). The second statement is a minor modification of Theorem 4.

In order to realize the programme [IŁ4] (2) of the first of us, we have still to deform the G_k and $\mathcal{G}^k$-structures and therefore to go outside of SU(2) and su(2).

15. The case of an arbitrary symmetry within $\mathbf{SO}(r+1,s)$

We make two simple, but important observations; cf. [Bl, Bo, CJ], [AŁ], pp. 45–46, and [W], pp. 69 and 78:

Remark 2. Theorem 5 extends for an arbitrary compact subgroup of $\mathrm{SO}(r_k+1,s_k)$, $\mathrm{SO(m)}$ or $\mathrm{SU(m)}$.

Remark 3. (i) Suppose that $\mathbf{E}_h=(E_h,\pi_h,M_h)$ is a real G_k-vector bundle over M_h and N_h is a given G_h-invariant connection corresponding to a $\mathcal{G}^h$-vector field, where G_h is an arbitrary compact subgroup of $\mathrm{SO}(r_h+1,s_h)$ or $\mathrm{SO(m)}$. Let $U_h=U_j^h$, $j\in J_h$, be a covering of M_h with local frames over U_j such that the corresponding transition matrices have their values in G_h. Then on E_h there exists a unique remannian metric g^e such that every above described system of local frames

$$\left(\varepsilon_{hj}^{\mu}\right),\quad \mu=1,\dots,\mathrm{m}\ \text{ with }\ \left(\pi_h\circ\varepsilon_{hj}^{\mu}\right)=\mathrm{id}_{U_j^h},\quad j\in J_h,\quad \mathrm{m}=\dim E_h, \tag{86}$$

satisfies the conditions

$$g^e\left(\varepsilon_{hj}^{\lambda},\varepsilon_{hj}^{\mu}\right)=\delta^{\lambda\mu}\quad\text{(the Kronecker symbol)},\quad \lambda,\mu=1,\dots,m. \tag{87}$$

The metric g^e satisfies the differential equation

$$dg^e(s,t)=g^e(N_hs,t)+g^e(s,N_ht),\quad s,t\in\Gamma_h(E_hM_h), \tag{88}$$

with $\Gamma_h(E_h,M_h)$ being the modulus of cross-sections of E_h over M_h. Therefore the metric g^e determines a class of real G-vector bundles M_e and G_e-invariant connections N_e with M_e dual to M_h in the sense of Theorem 5.

(ii) Conversely, for every riemannian metric g_e of E_e in a G_e-vector bundle $\mathbf{E}_e=(E_e,\pi_e,M_e)$ on E_e there exists a unique G_h-structure determined as the reductions of the bundle of orthonormal bases for E_h. This determines a class of G_k-invariant connections N_h corresponding to the $\mathcal{G}^h$-vector fields and, as a consequence, a class of real G_h-vector bundles M_h with M_h dual to M_e in the sense of Theorem 5. The whole remark is still correct if we interchange the members of one or more pairs of notations and adjectives as listed in Table 3.

Tab. 3. Admissible pairwise alterations in Remark 3

Allowed	↓	interchange
indices e together with the adjectives "riemannian"	⟷	indices h together with the adjectives "pseudoriemannian"
$\mathrm{SO}(r_k+1,s_k)$ or $\mathrm{SO(m)}$	⟷	$\mathrm{SU(m)}$
real	⟷	complex
"riemannian" or "pseudoriemannian"	⟷	hermitian
orthonormal	⟷	unitary

It is unknown under which conditions, starting with a flat space-time M_h, we can arrive at a curved Riemannian manifold M_e with solenoidal connection N_e.

16. Simplifying the external field in terms of the metric and connection

We are prepared to prove now, after [ŁKS], the theorem on simplifying the external field in terms of the metric and connection. This is crucial for realizing the scheme [IŁ4] (2) of the first of us, aiming at simplifying deformation of the external field:

> the potential V replaced by $V_\#$ (notation as in the quoted formula) and replacing this by a more sophisticated, deformed geometry:
>
> topological structure of the space M_0 replaced by $M_\#$ (e.g. multiply connected), metric F_0 replaced by $F_\#$; connection N or N^C replaced by $N_\#$.

We remark that, because of the already quoted (at least in Sect. 13, before giving the definition of a solenoidal connection) ideas of passing from M_0 to an analogue of Fischer's information or to the associated Riemannian space, as well as because of Remark 3 giving a procedure of passing from the h-case to the e-case, we are concentrating on the riemannian metric g^e only.

In the case of external fields, according to their symmetries, the corresponding *generalized Yang-Mills equation* $D_k(\star_{\mathcal{G}^k}\Omega_k^2) = 0$, according to (85), has to contain the term $\star_{\mathcal{G}^k}\Omega_k^2 \wedge_{\mathcal{G}^k} \omega_k \neq 0$, $k = e$ or $k = h$. Consider the resolution

$$\star_{\mathcal{G}^k}\Omega_k^2 \wedge_{\mathcal{G}^k} \omega = \sum_\ell \sum_\alpha R_h^{\ell\alpha}\sigma_\ell^h dx_h^\alpha \text{ with } \Omega_h^2 = \sum_\ell \sum_\alpha \sum_\beta F_h^{\ell\alpha\beta}\sigma_\ell^h dx_h^\alpha dx_h^\beta$$
$$\text{and} \quad \omega_h = \sum_\ell \sum_\alpha A_{h\alpha}^\ell \sigma_\ell^h dx_h^\alpha,\ \ell = 1, \dots, \alpha - 1,$$

where (x^α) is a local co-ordinate system in S_h. In terms of the Pauli-type matrices σ_α is related in a known way to the generators of the Clifford algebra $Cl(r-1, s)$, $r = r_e$, $s = s_e$. Then $R_h^{\ell\alpha}$ appear to be complex-valued functionals depending, in general on all $F_h^{\ell\alpha\beta}, A_{h\alpha}^\ell$, and the *pseudoriemannian* metric g_k. These functionals are determined by the relations $[\sigma_j^h, \sigma_{j'}^h] = \sigma_\ell^h] = \sigma_\ell^h$ with distinct j, j', ℓ. Hence we deduce the appearance of the current $\mathbf{j} = \Box_h A_h \neq 0$ with the usual meaning of $\Box$. Clearly, $\mathrm{Div}_h A_h = 0$ and $\mathrm{Div}_e A_e = 0$. Therefore we have proved [ŁKS]:

Theorem 6. *Consider a pair of dual vector bundles*

$$\mathbf{E}_k = (E_k, \pi_k, M_k), \quad k = e \quad \textit{and} \quad k = h,$$

constructed in Remark 3. Suppose that $\Box_e \mathbf{A}_e = 0$. *Then there exists on* E_h *a unique metric* g_e *such that:*

(i) *every system of local frames* (86)*, over a member of a covering of* M_h *with the corresponding transition matrices having their values in* G_k*, satisfies the conditions* (87)*;*

(ii) g_e *satisfies the differential equation* (88)*, where* $\Gamma_h(E_h, M_h)$ *stands for the modulus of cross-sections of* E_h *over* M_h*. Moreover,* g_e, A_e *and* A_h *satisfy the differential equations*

$$d_e\left(\star_{\mathcal{G}^e}\Omega_e^2\right)=0,\ \operatorname{div}_e\mathbf{A}_e=0,\quad \textit{and}\quad D_h\left(\star_{\mathcal{G}^e}\Omega_e^2\right)=0,\quad \operatorname{Div}_e\mathbf{A}_h=0 \tag{89}$$

together with $\Box_h\mathbf{A}_h=\mathbf{j}$*, where* $\mathbf{j}$ *is of class* $C^\infty(M_h)$*. If the correspondence of* $\mathbf{A}_e^\ell$ *and* $\mathbf{A}_h^\ell, \ell\in I_h$ (cf. [RR] and [GrH], pp. 282–297) *is known, the equations*

$$\Box_e\mathbf{A}_e=0\quad \textit{and}\quad \Box_h A_h^\ell(\mathbf{j})=\mathbf{j} \tag{90}$$

form, together with (89)*, a self-consistent system for determining* $\mathbf{j}$ *in* M_h *so that the vector bundle* $\mathbf{E}_h$ *admits its dual* $\mathbf{E}_e$ *with a* G_e*-invariant connection* N_e *corresponding to* $\mathbf{A}_e$*.*

Therefore we conclude:

Corollary 2. *Under suitable conditions we can replace the initial field* A_h *with currents by a fictitious field* A_e *without them, compensating this by a sitable deformation of the vector bundle* $\mathbf{E}_h$ *to* $\mathbf{E}_e$*, especially the metric* g_h *to* g_e*;* cf. [ŁW1, 2, Ł1, ŁKW].

17. The Einstein centrifuge

Consider a frame of reference which steadily rotates with the angular speed ω in the Minkowski space-time round the x_3-axis. In the cylinder co-ordinate system $(\theta,\rho,\varphi,\zeta)$ of the moving frame we obtain the proper arc length element

$$ds^2=-\left[1-(\omega/c)^2\rho^2\right]d\theta^2+2\omega\rho^2 d\varphi d\theta+d\rho^2+\rho^2 d\varphi^2+d\zeta^2;$$

cf. e.g. [LL], Sect. 89. Hence it seems natural to take as a relatively simple example the metric

$$g^h=\left(g^h_{\alpha\beta}\right)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1+(\omega/c)^2(x^2+y^2)\end{pmatrix}$$

with (t,x,y,z) as the co-ordinates. Consequently, the equation $\Box_h\mathbf{A}_h=0$ becomes

$$\frac{(\omega/c)^2}{1-(\omega/c)^2(x^2+Y^2)}\left[(x^2+y^2)\frac{\partial^2}{\partial(ct)^2}A^e_\alpha+x\frac{\partial}{\partial x}A^e_\alpha+y\frac{\partial}{\partial y}A^e_\alpha\right]=j_\alpha,\quad \alpha=1,2,3.$$

With

$$A^e_\alpha=\frac{1}{4\pi}\int\limits_{M_h}\frac{1}{|r-r'|}j_\alpha(r',x^\alpha_h)d^3r',$$

we finally obtain

$$\frac{(\omega/c)^2}{1-(\omega/c)^2(x^2+y^2)}\left\{(x^2+y^2)\int\limits_{M_h}\frac{\partial^2}{\partial(ct)^2}\frac{j_\alpha(r',x^\alpha)}{|r-r'|}d^2r'\right.$$
$$\left.-\frac{1}{[1-(\omega/c)^2(x^2+y^2)]^2}\left[x\int\limits_{M_h}\frac{\partial}{\partial x}\frac{j_\alpha(r',x^\alpha)}{|r-r'|}d^3r'+y\int\limits_{M_h}\frac{\partial}{\partial y}\frac{j_\alpha(r',x^\alpha)}{|r-r'|}d^3r'\right]\right\}$$
$$=4\pi j_\alpha.$$

18. Physical conclusions

We conclude that the outlined Finsler-geometrical model of quantum electrodynamics is quite close to that for thermodynamics [Ł6] and is strongly motivated by the fact that, in general, we deal at the same time with more than one type of field and more than one type of external potential. Because of the ideas of Beil [Be1, 2] and Kerner [K, KBG], the model is also quite close to the five-dimensional approach, Kałuża-Klein theories, Clifford analysis by Hurwitz pairs, involving the need of solitons and fractals. One can see this in our study of ferroelectric crystals in a Finsler geometry and in the Finsler-geometrical counterpart of the sine-Gordon equation for the surface.

As demanded in the introduction, the model is directed to open and dissipative systems. Also, following the scheme [IŁ4] (2), we were looking for replacing of complicated potentials by more sophisticated metrics and nonlinear connections, that is, including an important part of external fields as an intrinsic part of the electrodynamical geometry constructed. Since in several cases, because of the concepts of Fisher's information and associated Riemannian space (cf. Sect. 13), it was sufficient to consider a Riemannian or pseudoriemannian metric, starting from [IŁ4] Sect. 2, we have discussed a generalized Dirac-Maxwell system, a useful complex-analytical approach related with establishing convolution equations, and then generalizing the staff to Yang-Mills systems in the presence of an external field. This included the case of an arbitrary symmetry within the groups SO(m) or SU(m), the global system, an SU(2)-based non-abelian generalization, and a generalization of the Lagrangian and its embedding in the electroweak model.

The distinction between solenoidal and nonsolenoidal parts of the generalized Yang-Mills equation (Sect. 13) led us to an important conclusion (Theorem 4) that if the principal fibre bundle involved is not trivial and admits a solenoidal connection, then the base space of the bundle is *multiply connected*. This corresponds perfectly to the fundamental papers by Misner and Wheeler [MiW], Dirac [D], and Sakharov [Sa], whose concept relied upon an assumption of multiple connectivity of the space in question. Because of the basic demand expressed in the initial scheme [IŁ4] (2), a special section is devoted to the Yang-Mills field vs. the Yang-Mills field itself,

stressing the duality between the Yang-Mills generation (of Hurwitz pairs) as seen by an observer (the hyperbolic cases) and as related to the centre of the system (the elliptic cases); in other words: the duality between the Yang-Mills generation as related to the space of observation and as related to the space associated with the centre of the system.

A study of arbitrary symmetries within the groups $\mathrm{SO}(r+1,s)$ allowed us to elaborate a procedure of transforming more complicated hyperbolic systems to simpler elliptic systems. The related conditions enabling a simplification of the external field in terms of the metric and nonlinear connection were expressed in Theorem 6. Its physical content was formulated in Corrolary 2. In other words it says that *under suitable conditions we can replace the initial field A_h with currents, related to the space of observations, by a fictitious field A_e without them, related to the space associated with the centre of the system, compensating this by suitable deformations of the metric and nonlinear connection.*

Going deeper, we may explain the magnetic moment of elementary objects in our theory by a suitable choice of one pseudo-riemannian manifold – *the space of observations* and two general Riemannian manifolds – the *spaces of the particle* connected with the external electromagnetic and nuclear fields, respectively [ŁW1, Wi1–3, Bu, Sa], completed by suitable Finsler structures and nonlinear connections. By a general Riemannian manifold we understand a Riemannian manifold whose associated tensor field is allowed to be degenerate. In this way the mass of a particle as well as its electromagnetic and even nuclear properties are determined by means as manifolds and mappings between the corresponding Hilbert spaces. A nuclear reaction is then to be interpreted as a mapping between the corresponding pseudo-riemannian manifolds and the associated Riemannian manifolds.

The proposal in question is competitive to the quantum field theory and presents a different way of describing the properties of physical objects. As already noticed, it is not the first trial in this direction; cf. [D, Wi1, 2, Bu, Sa]. The distinction between the space of observations and the space of the particle is in fact motivated by Dirac's considerations [D], but it seems that the paper [ŁW1] as well as the present constructions permit to include effectively electromagnetic and nuclear interactions.

As far as elementary particles are concerned, a further step is related with an almost complex manifold approach [ŁW2] which leads to relations between the curvature form of an almost complex manifolds, accounting for the symmetry classification schemes within the frame of principal fibre bundles, and a curved Minkowski spacetime via induced smooth mappings characterizing nuclear reactions of type

$$\mathrm{N} + \pi \rightleftarrows \mathrm{N},$$

where N is some nucleon and π the virtual π-meson of this reaction. Both approaches follow the same main idea of Wheeler [Wh] developed in a different way by Sakharov [Sa].

Following the papers [ŁW1, 2] its natural continuation [ŁKW] is dealing with the construction of fields connected with particles and regarded as a deformation of the

particle space, thus providing a natural continuation also of the paper [ŁW3] (already referred to in [IŁ4], Section 4), where a complex-analytical method of solving the generalized Dirac-Maxwell system was proposed for a certain class of complex-Riemannian metrics. In [ŁKW] an explicit calculation based on linearization of the spinor connections is given. By analogy to the general relativity, where the gravitational field and the space curvature are tightly related, the appearance of a particle determines the space geometry whose properties reflect the particle properties and describe the fields produced by them.

As stressed by Asanov [As], Chapter 5, as soon as the Finslerian structure reflecting the internal symmetry of a certain class of physical fields is found, we are able to describe both the gravitational and Yang-Mills fields. When deriving the field equations in the context of the gauge approach [IŁ1], the Finslerian (in particular, Randersian) technique offers a simple possibility of using the projection factors of the indicatrix to construct a Lagrangian linear in a gauge strength tensor as is done in the usual gravitational theory, whereas in the usual gauge approaches the simplest gauge field Lagrangian is quadratic in the gauge tensor. Finslerian geometrization of isotopic invariance is used.

Asanov clearly illustrates the relationship between the curvature tensor associated with a metric tensor and the Yang-Mills tensors. He concludes that there are no serious difficulties in reinterpreting Finsler geometry in precise mathematical terms as providing a geometrical basis for constructing theories for physical fields exhibiting internal symmetries. The exposition involves the general idea of parametrical representation of physical fields.

For further reading in this direction we recommend the monographs by Asanov [As] and Manin [Ma1, 2] as well as important papers [To, BPZ].

Acknowledgment

This work (J. Ł.) was partially supported by the Ministry of Sciences and Higher Education grant PB1 P03A 001 26 (Sections 10–11 of the paper) and partially by the grant of the University of Łódź no. 505/692 (Sections 12–18).

References

[AŁR] J. Adem, J. Ławrynowicz, and J. Rembieliński, *Generalized Hurwitz maps of the type* $S \times V \to W$, (a) Dep. de Mat. Centro de Investigación y de Estudios Avanzados preprint no. 80 (1990), ii + 20 pp., (b) Rep. Math. Phys. **37** (1996), 325–336.

[As] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theories* (Fundamental Theories of Physics 12), Reidel, Dordrecht 1985.

[At] M. F. Atiyah, *Geometry of Yang-Mills Fields* (Lezioni Fermiane), Accademia Nazionale dei Lincei-Scuola Normale Superiore, Pisa 1979.

[BPZ] L. Brizhik, B. Piette, and W. J. Zakrzewski, *Localised electron states in a modified nonlinear Schrödinger equation*, in: New Developments in Fundamental Interaction Theories, ed. by J. Lukierski and J. Rembieliński (ATP Conference Proceedings 589), American Institute of Physics, Melville, NY 2001, pp. 405–415.

[Be1] R. G. Beil, *Finsler and Kałuża-Klein gauge theories*, Internat. J. Theor. Phys. **32** (1933), 1021–1031.

[Be2] —, *Moving frame transport and gauge transformations*, Found. Phys. **25** (1995), 717–742.

[Bl] D. Blecher, *Physics from the G-bundle viewpoint*, Internat. J. Theor. Phys. **23** (1984), 735–751.

[Bo] J. R. Bourgignon, *Une stratification de l'espace des structures riemanniennes*, Compositio Math. **30** (1975), 1–41.

[BoL] —, H. B. Lawson, and J. Simons, *Stability and gap phenomena for Yang-Mills fields*, Proc. Nat. Acad. Sci. U.S.A. **26** (1979), 1550–1553.

[Bu] P. Burcev, *Structural models of electron and nucleon in general relativity*, Czechoslovak J. Phys. B **23** (1973), 1172–1180.

[CJ] R. Coqueraux and A. Jadczyk, *Symmetries of Einstein-Yang-Mills fields and dimensional reduction*, Commun. Math. Phys. **98** (1985), 79–104.

[D] P. A. M. Dirac, *The motion of an extended particle in the gravitational field*, in: Proc. Internat. Conf. on Relativistic Theories of Gravitation, Warsaw and Jabłonna 1962. Ed. by L. Infeld, Pergamon Press, Gauthier-Villars and PWN [Polish Scientific Publishers], Warsaw 1964, pp. 163–175.

[DZ] A. M. Din and W. J. Zakrzewski, *Further properties of classical solutions in grassmannian σ-models*, Nuclear Physics B **237** (1984), 461–476.

[GH] S. G. Gindikin and G. M. Henkin, *Penrose's transformation and complex integral geometry* [in Russian], in: Itogi nauki i techniki. Seriya: Sovremennye problemy matematiki 17, Moskva 1981, pp. 57–111.

[GKŁ] B. Gaveau, J. Kalina, J. Ławrynowicz, and L. Wojtczak, *The global Yabg-Mills equations depending on an arbitrary metric*, Ann. Polon. Math. **46** (1985), 105–114.

[GKaŁ] —, —, —, P. Walczak, and L. Wojtczak, *On solenoidal Yang-Mills fields*, in: Complex Analysis and Applications, Varna 1983, Proceedings. Ed. by L. Iliev and I. Ramadanov, Publ. House of the Bulgarian Acad. of Sciences, Sofia 1985, pp. 81–86.

[GŁW2] —, J. Ławrynowicz, and L. Wojtczak, *Statistical mechanics of anharmonic crystals*, Physica Status Solidi (b) **121** (1984), 47–58.

[GrH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, J. Wiley and Sons, New York-Chichester-Brisbane-Toronto 1978.

[Hu] F. Hund, *Geschichte der physikalischen Begriffe*, Nachdruck der Ausg. von 1989 (Teil I) und 1987 (Teil II), Spektrum Akad. Verlag, Heidelberg-Berilin-Oxford 1996.

[IŁ1] R. S. Ingarden and J. Ławrynowicz, *Randers geometry and gauge theories II. Randers and Kałuża-Klein gauge theories, self-duality and homogeneity*, Bull. Soc. Sci. Lettres Łódź **58** Sér. Rech. Déform. **56** (2009), 49–60.

[IŁ4] —, —, *Finsler-geometrical model of quantum electrodynamics I. External field vs. Finsler geometry*, ibid. **60** Sér. Rech. Déform. **60**, no. 1 (2010), 155–174.

[K] R. Kerner, *Électrodynamique non linéaire en théorie de Kaliza et Klein*, C. R. Acad. Sci. Paris Série II **304** (1987), 621–624.

[KBG] —, A. L. Barbosa, and D. V. Gal'tsov, *Topics in Born-Infeld electrodynamics*, in: New Developments in Fundamental Interaction Theories, ed. by J. Lukierski and

J. Rembieliński (AIP Conference Proceedings 589), American Institute of Physics, Malville, NY 2001, pp. 377-389.

[KLŁ] W. Królikowski, D. Lambert, J. Ławrynowicz, and J. Rembieliński, *The* CR *problem and Hurwitz pairs applied to field theory and solid state physics*, (a) Dep. de Mat. Centro de Investigación y de Estudios Avanzados México Preprint no. 8 (1988), ii + 48 pp., (b) Rep. Math. Phys. **9** (1991), 23–44.

[KŁS] Katarzyna Kędzia, J.Ławrynowicz, and O. Suzuki, *Supercomplex structures, surface soliton equations, and quasicon formal mappings* (a) Dep. de Mat. Centro de Investigación y de Estudios Avanzados México Preprint no. 46 (1989), ii + 45 pp., (b) Ann. Polon. Math. **55** (1991), 245–268.

[LL] L. D. Landau and E. M. Lifschitz, (a) *Lehrbuch der theoretischen Physik*. Band *II. Klassische Feldtheorie*, 6th ed. [transl. from Russian], Akademie-Verlag, Berlin 1973; (b) *Course of Theoretical Physics*, Vol. *II . The Classical Theory of Fields* [transl. from Russian], Pergamon Press, London 1959.

[Ł1] J. Ławrynowicz, *Application à une théorie non linéaire de particules élémentaires*. Appendice pour le travail de B. Gaveau et J. Ławrynowicz: *Intégrale de Dirichlet sur une variété complexe. I*, in: Séminaire Pierre Lelong-Henri Skoda (Analyse), Années 1980/1981 (Lecture Notes in Mathematics 919), Springer, Berlin-Heidelberg 1982, pp. 152–166.

[Ł2] —, *Connections corresponding to hermitian structures* [*Problems in analysis on complex manifolds* (related to the Yang-Mills field)], in: Analytic Functions, Błażejewko 1982, Proceedings. Ed. by J. Ławrynowicz (Lecture Notes in Mathematics 1093), Springer, Berlin-Heidelberg 1983, pp. 488–490.

[Ł3] J. Ławrynowicz, *Quaternions and fractals vs. Finslerian geometry. A. Quaternions, noncommutativity, and C^*-algebras; B. Algebraic possibilities* (*quaternions, octonions, sedenions*)*; 1. Quaternion-based modifications of the Cayley-Dickson process*, Bull. Soc. Sci. Lettres Łódź **55** Sér. Rech. Déform. **47** (2005), 55–66, 67–82, and 83–98.

[Ł4] —, *Quaternions and fractals vs. Finslerian geometry. 2. Noncommutative hypercomplex analysis and C^*-algebras generated by isometries; 3. Graded fractal bundles related to quaternionic structures*, ibid. **55** Sér. Rech. Déform. **48** (2005), 47–63 and 65–82.

[Ł5] —, *Quaternions and fractals vs. Finslerian geometry. 4. Isospectral deformations in complex Finslerian quantum mechanics; 5. Clifford-analytical description of quantum-mechanical gauge theories; 6. Jordan algebras and special Jordan algebras*, ibid. **56** Sér. Rech. Déform. **49** (2006), 35–43, 45–60, and 61–77.

[Ł6] —, *Randers and Ingarden spaces with thermodynamical applications*, ibid. **59** Sér. Rech. Déform. **58** (2009), 117–134.

[ŁKS] —, S. Koshi, and O. Suzuki, *Dualities generated by the generalized Hurwitz problem and variation of the Yang-Mills field*, (a) Hokkaido Univ. Preprint Series in Math. no. 98 (1991), II + 17 pp., (b) Analysis, Geometry and Groups: A Riemann Legacy Volume. Ed. by H. M. Srivastava and Th. M. Rassias, Hadronic Press, Palm Harbor, FL, 1994, pp. 275–288.

[ŁKW] —, Katarzyna Kędzia, and L. Wojtczak, *Fields connected with particles in terms of complex-Riemannian geometry*, (a) Dep. de Mat. Centro de Investigación y de Estudios Avanzados Méxixo Preprint no. 16 (1989), ii + 47 pp., (b) Z. Naturforsch. **45a** (1990), 81–94.

[ŁR1] — and J. Rembieliński, *Pseudo-euclidean Hurwitz pairs and generalized Fueter equations*, (a) Inst. of Math. Polish Acad. Sci. Preprint no. 355 (1985), ii + 10 pp., (b) Clifford Algebras and Their Applications in Mathematical Physics, Proceedings, Canterbury 1985. Ed. by J. S. R. Chisholm and A. K. Common (NATO-AS) Series C: Mathematical and Physical Sciences 183), D. Reidel Publ. Co., Dordrecht-Boston-Lancaster-Tokyo 1986, pp. 39–48.

[ŁR2] J. Ławrynowicz and J. Rembieliński, *Pseudo-euclidean Hurwitz pairs and the Kałuża-Klein theories*, (a) Inst. of Phys. Univ. of Łódź Preprint no. 86–8 (1986), 26 pp., (b)J. Phys. A: Math. Gen. **20** (1987), 5831–5848.

[ŁW1] — and L. Wojtczak, *A concept of explaining the properties of elementary particles in terms of manifolds*, Z. Naturforsch. **29a** (1974), 1407–1417.

[ŁW2] —, —, *On an almost complex manifold approach to elementary particles*, ibid. **32a** (1977), 1215–1221.

[ŁW3] —, —, *A complex-analytical method of solving the generalized Dirac-Maxwell system for complex-Riemannians metric*, Rev. Roumaine Math. Pures Appl. **33** (1988), 89–95.

[Ma1] Yu. I. Manin, *Quantum Groups and Non-Commutative Geometry*, Centre de Recherches mathématiques, Univ. de Montréal, Montréal 1988.

[Ma2] —, *Gauge Field Theory and Complex Geometry* (Grundlehren der mathematischen Wissenschaften 289), Springer, Berlin-Heidelberg-New York-Paris-Tokyo 1989.

[MiW] C. W. Misner and J. A. Wheeler, *Classical physics as geometry*, Ann of Phys. **2** (1957), 525–603.

[PM1] J. Pouget and G. A. Maugin, *Solitons in electroacoustic interactions in ferroelectric crystals. I. Single solitons and domain walls*, Phys. Rev. B **30** (1984), 5306–5328.

[PM2] —, —, *Solitons in electroacoustic interactions in ferroelectric crystals. II. Interactions of solitons and radiations*, ibid. **31** (1985), 4633–4649.

[RR] G. Riera and R. Rodrigues, *Algebraic correspondences and Fuchsian groups*, in: Complex Geometry Seminar II. Ed. by R. Rodrigues and G. Riera, Univ. Tecn. F. Santa Maria, Valparaiso, Chile 1986, pp. 113–133.

[Sa] A. D. Sakharov, *Topological structures of elementary charges and the* CPT*-symmetry* [in Russian], in: Problems of Theoretical Physics. A memorial volume to I. E. Tamm, Publ. House "Nauka", Moscow 1972, pp. 242–247.

[To] I. T. Todorov, *Conformal covariant instanton solutions of Euclidean Yang-Mills equations* (*An elementary introduction*), in: III School of Elementary particles and High Energy Physics, Primorsko 1977, Bulgarian Acad. of Sci., Sofia 1978, pp. 131–182.

[W] R. D. Wells, Jr., *Differential Analysis on Complex Manifolds*, Prentice Hall, Englewood Cliffs, NJ 1973.

[Wh] D. A. Wheeler, *Geometrodynamics in the modern theory of gravitation*, Lecture held at the 5th Conf. on Gravitation, cf. Abstracts of that conf. ed. by V. A. Fock, Publ. Hause of the Tbilisi Univ., Tbilisi 1968.

[Wi1] F. J. Wiśniewski, *Über zweiatomige Molekeln*, Zeitschr. f. Phys. **37** (1926), 126–136.

[Wi2] —, *Die magnetische Suszeptibilitäten von* O_2 *und* N_2, ibid., **41** (1927), 385–394.

[Wi3] —, *Die Fortpflanzung des Lichtes durch fremde Kraftfelder. II*, ibid. **51** (1928), 443–450.

Institute of Physics
Nicolaus Copernicus University
Grudziądzka 5, PL-87-100 Toruń
Poland
e-mail: romp@phys.uni.torun.pl

Institute of Physics
Univeristy of Łódź
Pomorska 149/153, PL-90-236 Łódź
Institute of Mathematics
Polish Academy of Sciences
Łódź Branch, Banacha 22, PL-90-238 Łódź
Poland
e-mail: jlawryno@uni.lodz.pl

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on March 2, 2010

MODEL FINSLEROWSKO-GEOMETRYCZNY ELEKTRODYNAMIKI KWANTOWEJ II

INTERPRETACJA FIZYCZNA KONEKSJI SOLENOIDALNYCH I NIESOLENOIDALNYCH W KANONICZNYCH GŁÓWNYCH WIĄZKACH WŁÓKNISTYCH

S t r e s z c z e n i e

W pierwszej części pracy przeanalizowaliśmy różne fizyczne oczekiwania wskazując na znaczenie geometrii nieriemannowskich dla elektrodynamiki kwantowej. W tej części, po naszkicowaniu pięciowymiarowego modelu typu Kałuży-Kleina, rozważamy kryształy ferroelektryczne w geometrii Finslera z uwzględnieniem równania sinus-Gordona dla powierzchni. Z kolei, w uogólnionych równaniach Yanga-Millsa, wyróżniamy i dyskutujemy ich solenoidalne i niesolenoidalne części z uwzględnieniem, w duchu wyposażenia geometrii Finslera w stosowne koneksje nieliniowe, koneksji odpowiadających polu Yanga-Millsa przeciwstawionych samemu polu Yanga-Millsa. Następnie rozważamy przypadek dowolnej symetrii w grupie $\mathrm{SO}(r+1,s)$ oraz zagadnienie uproszczenia pola zewnętrznego w terminach metryki i koneksji. Pracę kończymy przykładem i przeglądem perspektyw w przyszłości.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2010 Vol. LX

Recherches sur les déformations no. 2

pp. 117–124

Roman Stanisław Ingarden and Julian Ławrynowicz

FINSLER GEOMETRY AND PHYSICS. PHYSICAL OVERVIEW

Summary

An analysis of regions of natural applicability in physics of Finsler geometry is given almost twenty years after appearance of a monograph in the subject by P. L. Antonelli, the first-named author of this paper, and late Professor M. Matsumoto (1993). In the monograph quoted a special attention was paid to geometrical optics in isotropic media, physiological optics, electron optics with a magnetic field, dissipative mechanics and thermodynamics. Now we observe the natural applicability also in the following directions: – **1.** Matsumoto, generalized Matsumoto, and Randers-Ingarden spaces in thermodynamics, – **2.** Randers-Ingarden spaces in electrodynamics, especially in the model of magnetic electron microscope including the scanning microscope, – **3.** Randers geometry and gauge theories, – **4.** Randers antisymmetric metric in the space-time of generally relativity, – **5.** Finsler-geometrical model of quantum electrodynamics.

0. After almost twenty years, when looking to [AIM], we can see the necessity of more stressing the variability of physical states, the probability character of the field amplitudes and the time instability of particles and of their *open systems* as "objects" of physical investigation. These systems are closely related with the physical character of the basic manifold on which the fields are defined. If the space of that manifold is filled with homogeneous, not necessarily isotropic matter, e.g. a transparent crystal, the geometry will change from Euclidean or pseudo-Euclidean to Finslerian or pseudo-Finslerian. It is then natural to distinguish at least five directions **1.**–**5.** of our scientific interests, listed in the Summary.

1. Physically, as far as the first direction (**1.**) is concerned, our task causes the following demands [SSI1, Ł]:

- Matsumoto spaces in thermodynamics.
- Generalized Matsumoto spaces in thermodynamics.

- Principles of thermodynamics including the electromagnetic effects.
- Randers and Ingarden spaces vs. openess and dissipativity of the system.
- Hyperbolicity, Minkowskian spaces, and parabolicity in thermodynamic geometry.
- Thermodynamic parameters and geometry in presence of the electromagnetic field.
- Statistical or stochastical thermodynamics?

The programme involves two kinds of Randers-Ingarden spaces, i.e. Randers spaces (M, F) [IŁ3], Section 9, equipped with a Lorentz nonlinear connection N or a Cartan nonlinear connection N^c. We are dealing with at least two kinds of information (entropy). The first one is the Shannon information of $\eta = 1$ or -1:

$$S_\eta = -\ln p - \ln(1-p), \quad 0 \le p \le 1,$$

where p is the probability of the value -1 of η. The second one, S_σ, is assumed to be the quantum information (entropy) of the medium composing our physical system. Using the temporal projection $d\sigma^2 = \tilde{\sigma} dt^2$ in relation with the space-time elements

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - \eta d\sigma^2,$$

we have

$$S_\sigma = \tilde{\sigma}/c.$$

Both $\tilde{\sigma}$ and c (the light velocity in vacuum) have the dimension of velocity, so S_σ is a pure number, as it should be for information or entropy in physics. If we use physical units by introducing the Boltzmann constant $K_{\mathrm{B}} \approx 1.3806503 \cdot 10^{-23}\,\mathrm{JK}^{-1}$, entropy is also a pure number, since joule (J) and Kelvin (K) can both be considered in the statistical or stochastical thermodynamics as different units of energy. It seems that entropy is the only fundamental physical unit which is dimensionless (a pure number).

2. In the second direction we have to study in detail the following particular problems [SSI2, IŁ5, 6]:

- Additional interactions between gravitational and electromagnetic forces.
- Properties of electromagnetic "lenses" including the torsion of electron trajectories.
- Generalizations of the Helmholtz-Lagrange law for an electric "lens".
- Importance of combining the focal length calculation with a non-Riemannian geometry.
- Construction of a Randers-type electromagnetic space.
- Immersion electromagnetic "lenses" in practice and in the constructed Randers-type electromagnetic space.

- Torsion-depending deformations within the electromagnetic spaces.
- Electromagnetic space of an electromagnetic microscope.
- Deformation of potentials in an electromagnetic space with the help of generating functions.
- "Lens"-thickness depending deformations in relation with the scanning microscope.
- Explicit formula for the focal length depending on the electromagnetic "lens" thickness.
- Potentials generating functions dependence vs. immersion electromagnetic "lenses" dependence.

The programme is in some sense a natural consequence of Einstein's formulation of the general relativity. Namely, he demanded an inhomogeneous and anisotropic metric of space, or – in other words – an anisotropic space. Already four years after Einstein's discovery Carathéodory [C2] and von Neumann [vN1] considered a point- and direction-dependent metric, and Finsler [Fi] formulated the problem in detail. Its further specification was due to Randers [Ra, IL3] who proposed a point- and direction-dependent metric, according to Maxwell's description of a magnetic electron microscope and a number of other problems in electrodynamics. This line of thinking has been adapted in our programme with a suggestion of using a slightly more sophisticated Randers-Ingarden spaces.

3. In the third direction we have to discuss [IL2]:

- Solitons in the Randersian physics.
- Complex Randersian physics vs. isospectral deformations.
- Complex gauge connections of interacted fields.
- Hurwitz pair description of gauge theories.
- Generations of Kałuża-Klein dualities vs. Kałuża-Klein-type theories.
- Self-duality equations for gauge theories.
- Homogeneity vs. gauge theories of the second order.

The programme refers to Kałuża-Klein gauge theories, natural e.g. for theories unifying electrodynamics and thermodynamics, and the corresponding gauge transformation due to an external field leads, in general, to a Randers metric. Now, the Randers geometrical approach to gauge theories yields the theory containing solitons of field equations. It is then natural to distinguish static, dynamic, probabilistic, and quantum Randers spaces including, because of space-time, complex Randers structures.

A different idea of connecting Finsler geometry with gauge theories is due to Asanov [As]. He gives a Finslerian representations of gauge fields and tensors, discusses gauge-covariant derivatives of spinors and isospinors, investigates linear gauge

transformations and Finslerian geometrization of isotopic invariance. The approach is related with interaction models for physical fields exhibiting internal properties, and with developing various gauge-geometric generalizations of the theory of Yang-Mills fields. It generates general gauge field equations associated with the curved internal space. In particular, Asanov formulates the variational principle for the parametrical gauge fields, gives general gauge-covariant physical field equations, a parametrical representation of the (x, y)-dependent gauge fields associated with the space-time, transition to the parametrical Finslerian limit, and characterizes proper Finslerian gauge transformations.

4. In the fourth direction we have to describe [IŁ4]:

- Forward and backward metrics in general relativity.
- Convexity of spheres and perpendicularity.
- Asymptotes and parallels.
- Axial motions and translations.
- Coincidence of the manifold topology with that generated by forward metric balls.
- The clocks synchronization.
- The interia tensor vs. non-inertial frames.
- Spin connections of the triple of correlations diffeomorphism ξ of two physical systems, ξ-morphism e of the related vector bundles, and the metric F_0.

In this part of the programme, in connection with the general relativity, and for showing the role of asymmetric, in particular antisymmetric metrics, we have to discuss the convexity of spheres, perpendicularity, asymptotes, parallels, axial motions, and translations in straight surfaces. In addition to these topics we have to explore the fact that the manifold topology coincides with that generated by forward metric balls. In the space-time of general relativity good examples of important problems are: the clocks synchronization, a strange behaviour of relativistic correlations vs. the quantum preferred frame, the inertia tensors vs. non-inertial frames, and spin correlations.

5. In the fifth direction we have to study [IŁ7, 8]:

- Generalized Dirac-Maxwell systems.
- Complex-analytical approach vs. physical demands.
- Related convolution equations with applications.
- A generalization: Yang-Mills equation in the presence of an external field.
- The case of an arbitrary symmetry within the groups SO(m) or SU(m).
- The global Yang-Mills system.

- SU(2)-based non-abelian generalizations.
- Generalizations of the Lagrangian and its embedding in the electroweak model.
- A concept of the five-dimensional model of nonlinear electrodynamics.
- Ferroelectric crystals in a Finsler geometry.
- The Finsler-geometrical counterpart of the sine-Gordon equation for the surface.
- The solenoidal and nonsolenoidal parts of the generalized Yang-Mills equations as observed on the canonical principal fibre bundle.
- Connections corresponding to the Yang-Mills field vs. the Yang-Mills field itself.
- The case of an arbitrary symmetry within $\mathrm{SO}(r+1,s)$.
- Simplifying the external field in terms of the metric and connection.
- The Einstein centrifuge.

When constructing a Finsler-geometrical model of quantum electrodynamics, we take into account that in quantum relativistic mechanics of electrons there are some physical constants, like Planck's constant h, light velocity c, and the mass and spin constants of an electron. In the usual technical units corresponding to the human scale of dimensions these constants are either very big, like c, or very small, like h. This naturally suggests the limiting cases of non-relativistic and non-quantum physics. If we take both these limits, we obtain the approximation given by the usual classical non-relativistic physics which has no more cosmic or microscopic constants, and therefore is parameter- or scale-independent, just as traditional logic or mathematics are [I].

In the present, more ambitious theory we go back to the ideas of [Wi1–3, HeJ, J1, 2; Kl] and [JW, D, Bu, Sa, Wh]. We can see that the concept of a particle as an independent and constant physical thing actually disappears in contradiction to the first quantization theory. Particles are created and annihilated as excitations of the whole field and the number of particles is, in general, not preserved. Only the total electric charge of the field is preserved as the difference of the positive and negative charges of positrons and electrons. Quantum partcles are usually entangled with the whole quantum field; they also interact with other fields: electromagnetic, gravitational, etc., and therefore are usually unstable and have no individuality, only some statistical properties, like antisymmetric statistics for fermions, and symmetric one for bosons.

Acknowledgment

This work (J. Ł.) was partially supported by the Ministry of Sciences and Higher Education grant PB1 P03A 001 26 (Sections 0–2 of the paper) and partially by the grant of the University of Łódź no. 505/692 (Sections 3–5).

References

[AIM] P. L. Antonelli, R. S. Ingarden, and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Academic, Dordrecht 1993.

[As] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theories* (Fundamental Theories of Physics 12), Reidel, Dordrecht 1985.

[Bu] P. Burcev, *Structural models of electron and nucleon in general relativity*, Czechoslovak J. Phys. B **23** (1973), 1172–1180.

[C1] C. Carathéodory, *Über das Schwarzsche Lemma bei analytischen Funktionen von zwei komplexen Veränderlichen*, Math. Ann. **97** (1926), 76–98.

[C2] —, *Über eine spezielle Metrik, die in der Theorie der analytischen Funktionen auftritt*, Atti Pontificia Acad. Nuovi Lincei **80** (1927), 135–141.

[D] P. A. M. Dirac, *The quantum theory of electron*, Proc. Roy. Soc. (London) A **117** (1928), 610–624.

[Fi] P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen*, Dissertation, Göttingen 1918; 2nd ed. Birkhäuser, Basel 1951.

[He] W. Heisenberg und P. Jordan, *Anwendung der Quantenmechanik auf das Problem der anomalen Zeemaneffekte*, Zeitschr. f. Phys. **37** (1926), 263–277.

[I] R. S. Ingarden, *Open systems and consciousness. A physical discussion*, Open Systems and Information Dynamics **9** (2002), 339–369.

[IŁ1] — and J. Ławrynowicz, *Physical and biophysical interpretations of a new model of open systems with superluminal velocities*, Acta Physicae Superficierum **6** (2004), 91–106.

[IŁ2] —, —, *Randers geometry and gauge theories II. Randers and Kałuża-Klein gauge theories, self-duality and homogeneity*, Bull. Soc. Sci. Lettres Łódź **58** Sér. Rech. Déform. **56** (2009), 49–60.

[IŁ3] —, —, *Finsler geometry and physics. Mathematical overview*, ibid. **58** Sér. Rech. Déform. **57** (2008), 51–56.

[IŁ4] —, —, *Randers antisymmetric metric in the space-time of general relativity I. Outline of the method; II. Perspectives of applications*, ibid. **59** Sér. Rech. Déform. **59**, no. 1 (2009), 117–130 and 131–138.

[IŁ5] —, —, *Model of magnetic electron microscope including the scanning microscope I. Physical model; II. Mathematical model*, ibid. **59** Sér. Rech. Déform. **59**, no. 2 (2009), 107–117 and 119–126.

[IŁ6] —, —, *Model of magnetic electron microscope including the scanning microscope III. Variational approach and calculation of the focal length in a Randers-type geometry*, ibid. **60** Sér. Rech. Déform. **60**, no. 1 (2010), 137–154.

[IŁ7] —, —, *Finsler-geometrical model of quantum electrodynamics I. External field vs. Finsler geometry*, ibid. **60** Sér. Rech. Déform. **60**, no. 1 (2010), 155–174.

[IŁ8] —, —, *Finsler-geometrical model of quantum electrodynamics II. Physical interpretation of solenoidal and nonsolenoidal connections on the canonical principal fibre bundles*, ibid. **60** Sér. Rech. Déform. **60**, no. 2 (2010), 93–115.

[J1] P. Jordan, *Bemerkung über einen Zusammenhang zwischen Duanes Quantentheorie der Interferenz und den de Broglieschen Wellen*, Zeitschr. f. Phys. **37** (1926), 376–382.

[J2] —, *Über kanonische Transformationen in der Quantenmechanik*, ibid. **37** (1926), 383–386.

[JW] — und E. Wigner, *Über das Paulische Äquivalenzverbot*, ibid. **47** (1928), 631–651.

[Kl] O. Klein, *Quantentheorie und fünfdimensionale Relativitätstheorie*, Zeitschr. f. Phys. **37** (1926), 895–906.

[Ł] J. Ławrynowicz, *Randers and Ingarden spaces with thermodynamical applications*, Bull. Soc. Sci. Lettres Łódź **59** Sér. Rech. Déform. **58** (2009), 117–134.

[vN] J. von Neumann, *Thermodynamik quantummechanischer Gesamtheit*, Gött. Nachr. **1** (1927), 273–291; reprinted in his Collected Works, vol. I, Pergamon Press, Oxford-London-New York-Paris 1961, pp. 236–254.

[Ra] G. Randers, *On an asymmetric metric in the four-space of general relativity*, Phys. Rev. (2) **59** (1941), 195–199.

[SSI1] H. Shimada, S. V. Sabau, and R. S. Ingarden, *The* (α, β)*-metric for Finsler space and its role in thermodynamics*, Bull. Soc. Sci. Lettres Łódź **58** Sér. Rech. Déform. **57** (2008), 25–38.

[SSI2] —, —, —, *The Randers metric and its role in electrodynamics* ibid. **58** Sér. Rech. Déform. **57** (2008), 39–49.

[Sa] A. D. Sakharov, *Topological structures of elementary charges and the* CPT*-symmetry* [in Russian], in: Problems of Theoretical Physics. A memorial volume to I. E. Tamm, Publ. House "Nauka", Moscow 1972, pp. 242–247.

[Wh] D. A. Wheeler, *Geometrodynamics in the modern theory of gravitation*, Lecture held at the 5th Conf. on Gravitation, cf. Abstracts of that conf. ed. by V. A. Fock, Publ. Hause of the Tbilisi Univ., Tbilisi 1968.

[Wi1] F. J. Wiśniewski, *Über zweiatomige Molekeln*, Zeitschr. f. Phys. **37** (1926), 126–136.

[Wi2] —, *Die magnetische Suszeptibilitäten von* O_2 *und* N_2, ibid. **41** (1927), 385–394.

[Wi2] —, *Die Fortpflanzung des Lichtes durch fremde Kraftfelder. II*, ibid. **51** (1928), 443–450.

Institute of Physics
Nicolaus Copernicus University
Grudziądzka 5, PL-87-100 Toruń
Poland
e-mail: romp@phys.uni.torun.pl

Institute of Physics
Univeristy of Łódź
Pomorska 149/153, PL-90-236 Łódź
Institute of Mathematics
Polish Academy of Sciences
Łódź Branch, Banacha 22, PL-90-238 Łódź
Poland
e-mail: jlawryno@uni.lodz.pl

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on July 16, 2010

GEOMETRIA FINSLERA A FIZYKA. SPOJRZENIE FIZYCZNE

S t r e s z c z e n i e

Niemal dwadzieścia lat po ukazaniu się w zakresie rozpatrywanego tematu monografii P. L. Antonelliego, pierwszego z autorów obecnej pracy i nieodżałowanej pamięci Profesora M. Matsumoto (1993) podajemy kolejną analizę naturalnych obszarów stosowalności geometrii Finslera w fizyce. W cytowanej monografii szczególna uwaga była zwrócona na

optykę geometryczną ośrodków niejednorodnych, optykę fizjologiczną, optykę elektronową z działaniem pola magnetycznego, mechanikę dyssypatywną oraz termodynamikę. Obecnie obserwujemy naturalne zastosowania geometrii Finslera również w następujących kierunkach: – **1.** Przestrzenie Matsumoto, uogólnione przestrzenie Matsumoto i przestrzenie Randersa-Ingardena w termodynamice; – **2.** Przestrzenie Randersa-Ingardena w elektrodynamice, w szczególności w modelu magnetycznego mikroskopu elektronowego uwzględniającym mikroskop skaningowy; – **3.** Geometria Randersa i teorie cechowania; – **4.** Antysymetryczna metryka Randersa w czasoprzestrzeni ogólnej teorii względności; – **5.** Model Finslerowsko-geometryczny elektrodynamiki kwantowej.

Rapporteurs – Referees

Richard A. Carhart (Chicago)
Fray de Landa Castillo Alvarado (México, D.F.)
Pierre Dolbeault (Paris)
Paweł Domański (Poznań)
Mohamed Saladin El Nashie (London)
Jerzy Grzybowski (Poznań)
Ryszard Jajte (Łódź)
Zbigniew Jakubowski (Łódź)
Tomasz Kapitaniak (Łódź)
Grzegorz Karwasz (Toruń)
Leopold Koczan (Lublin)
Ralitza K. Kovacheva (Sofia)
László Kozma (Debrecen)
Dominique Lambert (Namur)
Andrzej Łuczak (Łódź)
Cecylia Malinowska-Adamska (Łódź)
Stefano Marchiafava (Roma)
Andrzej Michalski (Lublin)
Leon Mikołajczyk (Łódź)
Marek Moneta (Łódź)
Yuval Ne'eman (Haifa)
Tatsuro Ogata (Tokyo)
Dariusz Partyka (Lublin)
Adam Paszkiewicz (Łódź)
Krzysztof Podlaski (Łódź)
Yaroslav G. Prytula (Kyiv)
Henryk Puszkarski (Poznań)
Monica Roşiu (Craiova)
Jakub Rembieliński (Łódź)
Stanisław Romanowski (Łódź)
Ken-Ichi Sakan (Osaka)
Hideo Shimada (Sapporo)
Józef Siciak (Kraków)
Osamu Suzuki (Tokyo)
Józef Szudy (Toruń)
Luis Manuel Tovar Sánchez (México, D.F.)
Francesco Succi (Roma)
Anna Urbaniak-Kucharczyk (Łódź)
Włodzimierz Waliszewski (Łódź)
Grzegorz Wiatrowski (Łódź)
Władysław Wilczyński (Łódź)
Hassan Zahouani (Font Romeu)
Lawrence Zalcman (Ramat-Gan)
Natalia Zorič (Kyiv)

CONTENU DU VOLUME LX, no. 3