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TITLE – INSTRUCTION FOR AUTHORS SUBMITTING THE PAPERS FOR BULLETIN

Summary

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The paper for BULLETIN DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ should be written in LaTeX, preferably in LaTeX 2e, using the style (the file **bull.cls**).

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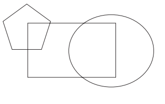

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References

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TABLE DES MATIÈRES

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Promarz Tamrazov
in memoriam
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Recherches sur les déformations no. 3

pp. 9–15

In memory of
Professor Promarz M. Tamrazov

Iwona Łużniak and Anna Urbaniak-Kucharczyk

SOME MAGNETIC PROPERTIES OF BILAYER WITH ROUGH AND DISORDERED INTERFACES

Summary

The interlayer exchange coupling (IEC) is an important property determining magnetic characteristics of multilayers. The influence of spacer thickness, roughness and structural ordering on interfaces on IEC have been shown on the basis of some theoretical predictions. The Green function method applied to the Hamiltonian of layered system allows to find Curie temperature, magnetization profiles and B parameter in $T^{3/2}$ Bloch's law.

Keywords and phrases: interlayer exchange coupling, roughness, Bloch's law

1. Introduction

Magnetic properties of bilayer consisting of two magnetic layers separated by a non-magnetic metal spacer have been studied both from theoretical and experimental point of view for many years. Especially, many papers were devoted to the interlayer exchange coupling (IEC) as a quantity determining the other magnetic characteristics. The possibility of technical applications of very thin layers and/or multilayers is the reason for introducing some parameters describing roughness and structural ordering in the interfaces in theoretical models of such systems. In this paper we take into consideration two types of parameters modifying the IEC describing the roughness and electron wave scattering.

The Green function method applied to the Hamiltonian of layered system in the first order approximation allows us to find some magnetic characteristics like Curie temperature, magnetization profiles and parameter B in the $T^{3/2}$ Bloch's law.

2. Theoretical model of the bilayer exchange coupled with rough interface system

We consider the bilayer consisting of two ferromagnetic layers and nonmagnetic spacer. The thickness of each magnetic layer equal to N and spacer - d is given by the number of monoatomic planes. We assume the Hamiltonian for this system as the sum of the exchange, Zeeman and anisotropy term in the form [18]:

$$\mathcal{H} = -\frac{1}{2}\sum_{\nu j \nu' j'} J_{\nu j \nu' j'}\mathbf{S}_{\nu j}\mathbf{S}_{\nu' j'} - g\mu_B H_{eff}\sum_{\nu j}\mathrm{S}^z_{\nu j} - \frac{1}{2}\sum_{\nu j \nu' j'} A_{\nu j \nu' j'}\mathbf{S}_{\nu j}\mathbf{S}_{\nu' j'}, \tag{1}$$

where $\nu \in (1, 2N)$ denotes the number of monatomic plane and j defines the position of a lattice point in the plane ν. $J_{\nu j \nu' j'}$ is exchange parameter equal to J_{12} for $\nu = N$ or $N+1$ and J in the other cases. The anisotropy parameter $A_{\nu j \nu' j'}$ consists of the uniaxial A, interface A_I and surface A_s anisotropy parameters, respectively. The effective field H_{eff} contains the external uniform field, the demagnetizing field and the uniaxial bulk anisotropy field. We applied the standard Green function procedure in the first order approximation described in [19] to such system and calculated Curie temperature, magnetization profiles and parameter B in Bloch's law $T^{3/2}$

$$m(T) = m_0(1 - BT^{3/2}). \tag{2}$$

3. Interlayer exchange coupling

In many theoretical models IEC can be calculated as the total energy difference between the parallel and antiparallel configurations of magnetization in both magnetic layers using *ab initio* method [1, 2], semi-empirical tight-binding scheme [3] and quantum well model [4–7]. These methods are difficult for numerical calculation because the difference of energy for different orientation of magnetization vector is several orders of magnitude smaller than the total energy of such system.

Other possibilities are given by model proposed by Bruno [8, 9] in the frame of scattering theory of electron wave or Bruno and Chappert [10, 11] based on generalization of RKKY scheme. We show calculation of IEC parameter for two last models considering the influence of structural disorder in the interface described by roughness and scattering parameters.

3.1. Roughness parameter

The generalization of the RKKY theory proposed by Bruno and Chappert [10, 11] gives the dependance of IEC on the spacer thickness in the form:

$$J_{12} = -\frac{1}{2}\left(\frac{\mathcal{A}}{V_0}\right)^2 \frac{V_0}{(2\pi)^3}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} f^2(q_z)\mathrm{d}q_z\chi(q_z)\,\exp(iq_z z). \tag{3}$$

$\chi(q_z)$ is the nonuniform susceptibility and depends on the material parameters of spacer and V_0 is the atomic volume.

The distribution function $f(q_z)$ plays an important role in the model introduced by Wang, Levy and Fray [12]. for ideal flat interfaces this function is equal to one and for rough interfaces takes the form

$$f(q_z) = \sum_{n=-\infty}^{\infty} p_n e^{iq_z na/2}. \tag{4}$$

p_n denotes the probability of finding a magnetic atom at distance n times $a/2$ away from the average position of the surface. In practice, we take that $p_1 = p_{-1} = r$ and $p_0 = 1-2r$ while all p_n for the other n are equal to 0. The choice of values for p_n must grant separation of magnetic layers in spite of possibility of finding magnetic atoms in the spacer material. This implies a minimal thickness of spacer layer. Parameter r changes from 0 for flat to 0.25 for rough interface and can be related to the root mean square *rms* [13,14] of deviation from perfectly flat interface by $r = 2\ rms^2/a^2$.

The other form of distribution function is given by Solid-on-Solid (SOS) and Discrete Gaussian (DG) model [15]. They are usually used for description of the dynamical growth of a layer and can be applied to the static case. In this formalism the layer is treated as a twodimensional matrix filled by atomic columns. The interaction between columns with hight h_i is done by the Hamiltonian [16]:

$$\mathcal{H} = J/2 \sum_i \sum_\delta (h_i - h_{i+\delta})^2 + y \sum_i (1 - \cos 2\pi h_i) - \Delta\mu \sum_i h_i, \tag{5}$$

where $\Delta\mu$ is chemical potential and y - weight which favors integer values of h_i.

The time evaluation of such system is described by the Fokker-Planck equation for the probability $P(\{h_i\}, t)$ of some configuration of $\{h_i\}$ in the moment t. Taking into consideration the correlation function G_{ij} we can show the probability $P(\{h_i\}, t)$ in the form:

$$P(\{h_i\}, t) = \exp\left[-\frac{1}{2} \sum_i \sum_j (h_i - h(t)) G_{ij}^{-}1(t)(h_j - h(t)) \right]. \tag{6}$$

The Fourier transformation of this formula gives us distribution function $f(q_z)$.

3.2. Effect of alloying

The roughness in the interface region can be treated as an intermixing of different atoms caused by an alloying process. Theoretical model proposed by Bruno [8] for the perfect flat interfaces can be adopted to the disordered alloy in the interface region. This model is based on the reflection and transmission amplitudes of electron wave. Disorder in the interface modifies a specular and diffuse scattering of electron in the interface [17]. The interlayer exchange coupling is proportional to the function

$$\begin{aligned} f(\alpha', \alpha''; k^F d) &= \frac{1}{(k^F d)^2} \{[(1-\alpha'')^2 - \alpha^2] \sin(2\mathrm{k^F d}) \\ &\quad + 2\alpha'(1-\alpha'')\cos(2\mathrm{k^F d})\}, \end{aligned} \tag{7}$$

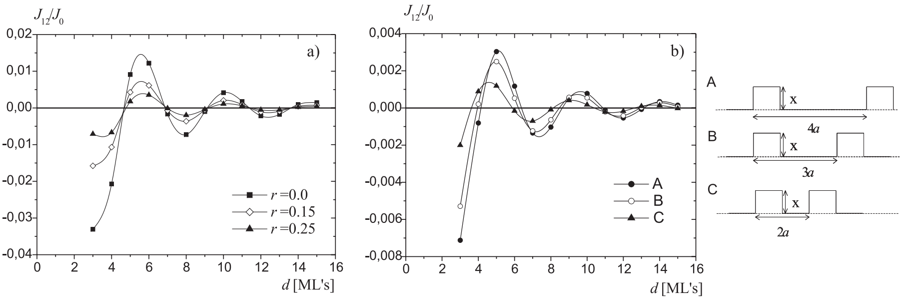

Fig. 1: The influence of a) roughness parameter r and b) roughness distribution in the interface on the interlayer exchange coupling in the function of spacer thickness d. r equal to 0 denotes the flat interface. Increasing of this value is related to a rougher interface.

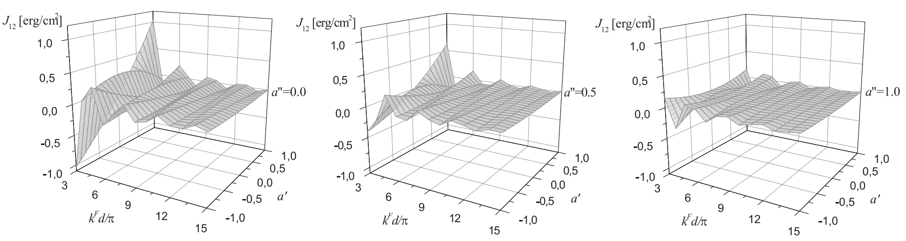

Fig. 2: The interlayer exchange coupling for chosen values of diffuse scattering parameter α'' as a function of specular scattering parameter α' and spacer thickness d.

where k^F denotes the Fermi wave vector. Two additional parameters α' and α'' are connected to the specular and diffuse scattering, respectively. They are a real and imaginary part of dimensionless complex parameter $\alpha = \alpha' + i\alpha''$ and it is convenient that $\alpha = 0$ denotes a perfect interface.

The dependence of IEC on the spacer thickness is connected to the value and sign of scattering parameters, which should be properly limited. Simple consideration leads to statement that α' may change sign and can be related to the concentration of magnetic atoms in the interface. From the physical point of view the positive value of α'' properly provides damping of IEC.

4. Numerical results

It is known from previous theoretical [1,4,5,20] and experimental [21–23,34] considerations that the interlayer exchange coupling is a decreasing and oscillating function of spacer thickness. We suppose that roughness is the damping parameter for the IEC. Its oscillation amplitude has a universal d^{-2} decay [25]. Results in the Fig. 1a

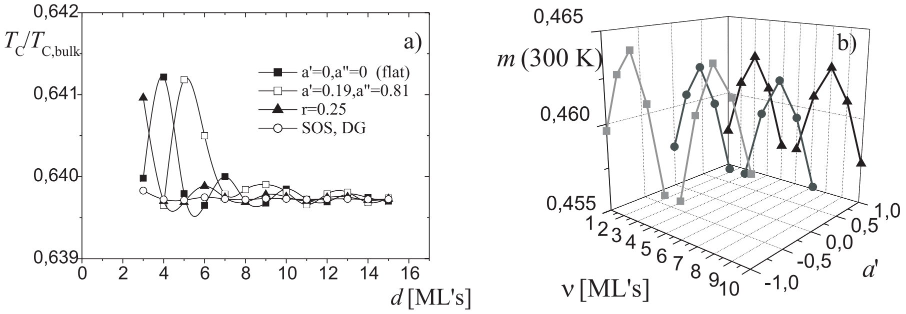

Fig. 3: The dependance of Curie temperature on the spacer thickness and different models of roughness (a) and (b) magnetisation profiles for α' equal to -0.5, 0.0 and 0.5.

presented for the model with r parameter confirm our expectation and give good values of period of oscillation in comparison to the theoretical and experimental results [26–31] but we can not observe a phase shift in the oscillation of IEC. It is necessary to notice that the negative values of IEC in this figure are related to the ferromagnetic interaction between magnetic layers. The results obtained in the frame of SOS and DG models confirm previous observations and additionally the small shift of the phase of oscillation is observed (Fig. 1b). Presented values of IEC are in good agreement with [32, 33] especially taking into consideration findings from [21, 34].

The interlayer exchange coupling calculated in the frame alloying model has similar oscillating character as was earlier mentioned. The phase of oscillation and phase shift is connected to the specular scattering parameter α'. The damping of IEC is related here to two reasons: change of spacer thickness and diffuse scattering. All those dependencies can be observed in Fig. 2, where the exchange coupling is presented as a function of spacer thickness d and specular scattering parameter for three different values of diffuse scattering parameter α''.

The IEC is an important property of a layered system. The oscillating and decreasing character can also be observed in behavior of other characteristics of such systems. In the Fig. 3 we present the comparison for Curie temperature calculated using the IEC from considered models. The influence of roughness and alloying in the interface on the value of Curie temperature is not significant but disordered interface shifts the phase of oscillation. As we expected the oscillation period is equal to half of period of IEC. Our results are in good agreement with results presented in [36, 37].

The same oscillating character with period equal to half of IEC oscillation period can be observed for magnetization of one chosen magnetic layer numbered by ν for different spacer thickness. As can be expected the damping of magnetization of the whole system in given temperature is connected with disorder in the interface. The more significant modification of magnetization profile is observed only in the interface region.

The parameter B from Bloch's low (2) may be very interesting from the experimental point of view. Using calculated IEC parameter we obtain that $B \sim 10^{-5}\mathrm{K}^{3/2}$ which is in good agreement with values achieved in experiment [38–40].

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Presented by Ilona Zasada at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 13, 2012

WŁASNOŚCI MAGNETYCZNE DWUWARSTW Z SZORSTKĄ I ROZPORZĄDKOWANĄ MIĘDZYWIERZCHNIĄ

S t r e s z c z e n i e

Międzywarstwowe sprzężenie wymienne jest istotnym czynnikiem określającym własności magnetyczne wielowarstw. W pracy, na bazie teoretycznych przewidywań pokazano wpływ grubości przekładki, szorstkości międzywierzchni oraz uporządkowania strukturalnego na zachowanie się międzywarstwowego sprzężenia wymiennego. Zastosowana w obliczeniach metoda funkcji Greena dla Hamiltonianu opisującego układ warstwowy pozwala wyznaczyć temperaturę Curie, profile namagnesowania oraz parametr B w prawie Blocha.

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Dedicated to the memory of
Professor Promarz M. Tamrazov

Andriy O. Kuryliak, Oleg B. Skaskiv, and Igor E. Chyzhykov

BAIRE CATEGORIES AND WIMAN'S INEQUALITY FOR THE ANALYTIC FUNCTIONS

Summary

Let $f(z)=\sum_{n=0}^{+\infty} a_n z^n$ $(z\in\mathbb{C})$ be an analytic function in the unit disk and f_t be an analytic function of the form $f_t(z)=\sum_{n=0}^{+\infty} a_n e^{i\theta_n t} z^n$, where $t\in\mathbb{R}$, $\theta_n\in\mathbb{N}$, and h be a positive continuous on $(0,1)$ function increasing to $+\infty$ and such that $\int_0^1 h(r)dr=+\infty$. We prove that if the sequence $(\theta_n)_{n\geq 0}$ satisfies the inequality

$$(*)\qquad \overline{\lim}_{n\to+\infty}\frac{1}{\ln n}\ln\frac{\theta_n}{\theta_{n+1}-\theta_n}\leq\delta\in[0,1/2),$$

then for every analytic functions f almost surely for t there exists a set $E=E(\delta,t)\subset(0,1)$ such that $\int_E h(r)dr<+\infty$ and

$$(**)\qquad \overline{\lim}_{\substack{r\to 1-0\\ r\notin E}}\frac{\ln M_f(r,t)-\ln\mu_f(r)}{2\ln h(r)+\ln\ln\{h(r)\mu_f(r)\}}\leq\frac{1+2\delta}{4+3\delta},$$

where $M_f(r,t)=\max\{|f_t(z)|\colon |z|=r\}$, $\mu_f(r)=\max\{|a_n|r^n\colon n\geq 0\}$ for $r\in[0,1)$.

Keywords and phrases: random analytic functions, Wiman-Valiron's type inequality, Baire categories

1. Introduction

Let H be the class of positive continuous on the interval $(0,1)$ increasing to $+\infty$ functions and such that $\int_0^1 h(r)dr=+\infty$.

For a measurable set $E \subset (0,1)$, the h-measure of E is defined by

$$h\text{-meas } (E) \overset{def}{=} \int_E h(r)dr,$$

where $h \in H$. It is clear that h-meas$((0,1)) = +\infty$.

Let f be an analytic function in the unit disc $\mathbb{D} = \{z\colon |z| < 1\}$ of the form

$$f(z) = \sum\nolimits_{n=0}^{+\infty} a_n z^n. \tag{1}$$

For analytic function f in the unit disc $\mathbb{D}$ and for $r \in (0,1)$ by

$$M_f(r) = \max\{|f(z)|\colon |z| = r\} \quad \text{and} \quad \mu_f(r) = \max\{|a_n|r^n\colon n \geq 0\}$$

we denote the maximum modulus and maximal term of the series (1) respectively. Denote also

$$G_f(r) = \sum\nolimits_{n=0}^{+\infty} |a_n|r^n, \ S_f(r) = \Big(\sum\nolimits_{n=0}^{+\infty} |a_n|^2 r^{2n}\Big)^{1/2},$$

$$\Delta_h(r,f) = \frac{\ln M_f(r) - \ln \mu_f(r)}{2\ln h(r) + \ln_2\{h(r)\mu_f(r)\}},$$

$$E(\eta, f, h) = \big\{r \in (0,1)\colon M_f(r) > \mu_f(r)(h^2(r)\ln\{h(r)\mu_f(r)\})^{\eta}\big\},$$

where $\ln_k x \overset{def}{=} \ln(\ln_{k-1} x)$ $(k \geq 2)$, $\ln_1 x \overset{def}{=} \ln x$.

From results in [1] follows that for $h(r) = (1-r)^{-1}$ and for any analytic function f in $\mathbb{D}$ of the form (1) there exists a set $E \subset (0,1)$ of finite logarithmic measure, i.e. h-meas$(E) < +\infty$ with the function $h(r) = (1-r)^{-1}$, such that

$$\overline{\lim}_{\substack{r \to 1-0 \\ r \notin E}} \Delta_h(r,f) \leq \frac{1}{2}. \tag{2}$$

In [2] the similar statement is proved with arbitrary function $h \in H$ for which either $\ln h(r) = O(\ln_2 G_f(r))$ or $\ln_2 G_f(r) = O(\ln h(r))$ $(r \to 1-0)$.

In [3] it is noted that the constant $1/2$ in the inequality (2) in general cannot be replaced by a smaller number. Indeed, if

$$g(z) = \sum_{n=1}^{+\infty} \exp\{\sqrt{n}\} z^n,$$

then for $h(r) = (1-r)^{-1}$ we have

$$\underline{\lim}_{r \to 1-0} \frac{M_g(r)}{h(r)\mu_g(r)\ln^{1/2}\{\mu_g(r)h(r)\}} \geq C > 0.$$

In connection with this the following question arises naturally: *how can one describe the "quantity" of those analytic functions, for which inequality (2) can be improved?*

In the article [4] it is proved that in some probability sense for "majority" of analytic functions the constant $1/2$ in the inequality (2) can be replaced by $1/4$. Similar statement is proved in [2] in reference to the inequality (2) with any function $h \in H$ described above.

At the same time, the classes of random analytic functions considered in [2], [4] don't include all analytic functions of the form

$$f_t(z) = \sum_{n=0}^{+\infty} a_n e^{i\theta_n t} z^n, \tag{3}$$

where $(\theta_n)_{n\geq 0}$ is an arbitrary sequence of nonnegative integers. Note that $f_0(z) \equiv f(z)$.

We suppose that the sequence $(\theta_n)_{n\geq 0}$ satisfies inequality

$$\frac{\theta_{n+1}}{\theta_n} \geq q > 1 \ (n \geq 0). \tag{4}$$

In the case of $q \geq 2$ analytic functions of the form (4) satisfies the conditions of theorems from [2], [4] mentioned above.

We also remark that the possibility of improvement of Wiman-Valiron's inequality for entire functions of the form (3) were considered earlier by M. Still [5] and P. Filevych [6] (see also [7]). A similar question for the class of entire functions of two variables was concidered in the papers [8], [9] and [10]. In [11] "quantity" those entire functions for which classical Wiman-Valiron's inequality can be improved, is described in sense Baire categories.

Here we consider formulated *question* in the class of analytic functions in $\mathbb{D}$ of the form (3). Proved theorems complement in this case theorems from [2,4] and are analogues of the statements from [5] and [11].

2. Auxiliary lemmas

We need Lemma 2 from [5] (see also [6]).

Lemma 2.1. [5] *If a sequence $(\theta_n)_{n\geq 0}$ satisfies the condition (4), then for all sequences $(a_n)_{n\geq 0}$, $a_n \in \mathbb{C}$, and all $\beta > 0, N \geq 0$ we have*

$$P_0\Big(\Big\{t \in [0, 2\pi] \colon \max_{0\leq\psi\leq 2\pi} \Big|\sum_{k=0}^{N} a_k e^{ik\psi} e^{i\theta_k t}\Big| \geq A_{\beta q} S_N \ln^{1/2} N\Big\}\Big) \leq N^{-\beta},$$

where $A_{\beta q}$ is a constant which depends only on β and q, $S_N = \sum_{n=0}^{N} |a_n|^2$, $P_0 = \frac{\mathfrak{m}}{2\pi}$, $\mathfrak{m}$ is Lebesgue measure on the real line.

Lemma 2.2. [4] *Let $k(r)$ be continuous increasing to $+\infty$ on $(0,1)$ function, open set $E \subset (0,1)$ such that there exist the sequence $0 < p_1 \leq \ldots \leq p_n \to 1$ $(n \to +\infty)$ outside E. Then there exists an sequence $0 < r_1 \leq \ldots \leq r_n \to 1$ $(n \to +\infty)$ such that for all $n \in \mathbb{N}$: 1) $r_n \notin E$; 2) $\ln k(r_n) \geq \frac{n}{2}$; 3) if $(r_n; r_{n+1}) \cap E \neq (r_n, r_{n+1})$, then $k(r_{n+1}) \leq ek(r_n)$.*

Lemma 2.3. [2] *Let $\varphi_1(x)$ and $\varphi_2(x)$ be positive continuous increasing to $[0,+\infty)$ such that*

$$\int_0^{+\infty}\frac{dx}{\varphi_i(x)}<+\infty \quad (i\in\{1,2\}),\quad h\in H$$

and

$$g_1(x)=\ln G_f(e^x)\ (x<0).$$

Then there exists a set $E\subset(0,1)$ such that h-meas$(E)<+\infty$ and for all $r\in(0,1)\backslash E$ we get

$$g_1''(\ln r)\le h(r)\varphi_2(h(r)\varphi_1(g_1(\ln r))).$$

We also denote

$$A(r)=g_1'(\ln r)=\frac{d\ln G_f(r)}{d\ln r}=\sum_{n=0}^{+\infty}\frac{n|a_n|r^n}{G_f(r)},$$

$$B^2(r)=g_1''(\ln r)=\sum_{n=0}^{+\infty}\frac{n^2|a_n|r^n}{G_f(r)}-A^2(r).$$

Lemma 2.4. *For $h\in H$ and all $\varepsilon>0$ there exists a set $E\subset(0,1)$ such that h-meas$(E)<+\infty$ and for all $r\in(0,1)\backslash E$ we have*

$$A(r)\le h(r)\ln\{h(r)\mu_f(r)\}\ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\},$$
$$B^2(r)\le h^{2+\varepsilon}(r)\ln\{h(r)\mu_f(r)\}\ln_2^{2+\varepsilon}\{h(r)\mu_f(r)\}.$$

Proof. Let $(\Omega,\mathcal{A},P)$ be a probability space which consists the discrete random variable ξ with distribution

$$P(\xi=n)=\frac{|a_n|e^{nx}}{G_f(e^x)}.$$

Then the mean $M\xi=g_1'(x)$ and the variance $D\xi=g_1''(x)$.

Let $x=\ln r<0$. Using Chebychev's inequality we get

$$P(|\xi-g_1'(x)|<\sqrt{2g_1''(x)}\ )\ge 1/2,$$

i.e.

$$g(x)\le 2\sum_{|n-g_1'(x)|<\sqrt{2g_1''(x)}}|a_n|e^{xn}\le$$

$$\le 2\mu_f(r)\sum_{|n-g_1'(x)|<\sqrt{2g_1''(x)}}1\le 2\mu_f(r)(2\sqrt{2g_1{}''(x)}+1). \tag{5}$$

For fixed $\varepsilon_1>0,\ \varepsilon_2>0$ we define

$$E_1=\{x<0\colon g_1''(x)>h(e^x)g_1{}'(x)(\ln g_1'(x))^{1+\varepsilon_1},\ g_1'(x)\ge 2\},$$
$$E_2=\{x<0\colon g_1'(x)>h(e^x)g_1(x)(\ln g_1(x))^{1+\varepsilon_2},\ g_1(x)\ge 2\}.$$

So,

$$\int_{E_1 \bigcup E_2} h(e^x)dx = \int_E \frac{h(r)}{r}dr < +\infty,$$

thus $\int_E h(r)dr < +\infty$, where E is the image of the set $E_1 \bigcup E_2$ by mapping $r = e^x$. Therefore, h-meas$E = \int_E h(r)dr < +\infty$. Then from (5) we obtain as $r \to 1-0$, $(r \notin E)$

$$g(\ln r) \leq 2\mu_f(r)\Big(2\sqrt{2}\sqrt{h(e^x)g_1'(x)\ln^{1+\varepsilon_1} g_1'(x)} + 1\Big) \leq$$

$$\leq 4\mu_f(r)\left(\sqrt{2h^2(e^x)g_1(x)\ln^{1+\varepsilon_2} g_1(x)}\ln^{\frac{1+\varepsilon_1}{2}}\Big\{h(e^x)g_1(x)\ln^{1+\varepsilon_2} g_1(x)\Big\}+1\right) \leq$$

$$\leq 6\mu_f(r)h(r)\sqrt{g_1(x)}\ln^{\frac{1+\varepsilon_2}{2}} g_1(x)\ln^{\frac{1}{2}+\varepsilon_1}\{h(r)g_1(x)\},$$

$$g_1(x) = \ln g(x) \leq \ln 6 + \ln\{h(r)\mu_f(r)\} + \ln g_1(x) + \ln_2\{h(r)\mu_f(r)\},$$

$$g_1(x) \leq 2\ln\{h(r)\mu_f(r)\}.$$

Now for $\delta > 2(\varepsilon_1 + \varepsilon_2)$ we have

$$G_f(r) \leq \mu_f(r)h(r)\ln^{1/2}\{h(r)\mu_f(r)\} \times$$

$$\times\big(\ln_2\{h(r)\mu_f(r)\}\ln\{h(r)\ln\{h(r)\mu_f(r)\}\}\big)^{\frac{1+\delta}{2}},$$

$$M_f(r) \leq G_f(r) \leq \mu_f(r)h(r)\ln^{1/2}\{h(r)\mu_f(r)\}\ln^{1/2+\delta} h(r)\ln_2^{1+\delta}\{h(r)\mu_f(r)\}, \tag{6}$$

$$g_1(x) = (1 + o(1))\ln\{h(r)\mu_f(r)\},\ r \to 1-0,\ (r \notin E).$$

If we choose in Lemma 2.3 $\varphi_i(x) = (x+2)\ln^{1+\varepsilon_0/2}(2+x)$, $i \in \{1,2\}$, then we get outside a set of finite h-measure

$$A(r) \leq h(r)\varphi(g_1(\ln r)) \leq h(r)g_1(\ln r)\ln^{1+\varepsilon_0} g_1(\ln r) \leq$$

$$\leq h(r)\ln\{h(r)\mu_f(r)\}\ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\},$$

$$B^2(r) \leq h(r)\varphi_2(h(r)\varphi_1(g_1(\ln r))) \leq$$

$$\leq h(r)h(r)\varphi_1(g_1(\ln r))\ln^{1+\varepsilon_0}(h(r)\varphi_1(g_1(\ln r))) \leq$$

$$\leq h^2(r)g_1(\ln r)\ln^{1+\varepsilon_0} g_1(\ln r)\ln^{1+\varepsilon_0}\{h(r)g_1(\ln r)\ln^{1+\varepsilon_0} g_1(\ln r)\} \leq$$

$$\leq h^{2+\varepsilon}(r)\ln^{1+\varepsilon}\{h(r)\mu_f(r)\}.$$

The Lemma is proved.

3. Classes of analytic functions in which the Wiman-Valiron type inequality (2) can be almost surely improved

In the sequel, the notion "almost surely" will be used in the sense that the corresponding property holds almost everywhere with respect to Lebesgue measure on the real line. Here we will prove the following theorem.

Theorem 3.1. *If $f(z,t)$ is an analytic function of the form* (3) *and a sequence $(\theta_n)_{n\geq 0}$ satisfies condition* (4)*, then for all $\delta > 0$ and almost surely for t there exists a set $E(\delta,t) \subset (0,1)$ such that h-meas$(E(\delta,t)) < +\infty$ and the maximum modulus*

$$M_f(r,t)=M_{f_t}(r)=\max_{|z|\le r}|f_t(z)|$$

satisfies the inequality

$$M_f(r,t)\le \mu_f(r)\sqrt{h(r)}\ln^{1/4}\{h(r)\mu_f(r)\}\ln^{3/4+\delta}h(r)\ln_2^{1+\delta}\{h(r)\mu_f(r)\}\tag{7}$$

for $r\in(0,1)\backslash E(\delta,t)$.

We note that from inequality (7) it follows that

$$\overline{\lim}_{\substack{r\to1-0\\ r\notin E}}\Delta_h(r,f_t)=\overline{\lim}_{r\to1-0}\frac{\ln M_f(r,t)-\ln\mu_f(r)}{2\ln h(r)+\ln_2\{h(r)\mu_f(r)\}}\le\frac14.\tag{8}$$

Proof. Let $(\Omega,\mathcal{A},P)$ be a probability space which consists a random variable $X=X(\omega)\colon\Omega\to\mathbb{Z}_+$ with the distribution $P(X=n)=|a_n|r^n/G_f(r)$. Using Markov's inequality for the random variable X with mean value $MX=A(r)$ we get

$$\sum_{n\ge C}\frac{|a_n|r^n}{G_f(r)}=P(X\ge C)\le\frac{MX}{C}=\frac{A(r)}{C}.$$

Let $C=C(r)=A(r)h(r)\ln^{1/2+\delta}\{h(r)\mu_f(r)\}$, $C_1(r)=h^2(r)\ln^2\{h(r)\mu_f(r)\}$. By Lemma 2.4 $C_1(r)>C(r)$ for $r\in(r_0,1)\backslash E$. Using (6) we have

$$\sum_{n\ge C_1(r)}|a_n|r^n\le\sum_{n\ge C(r)}|a_n|r^n\le\frac{A(r)G_f(r)}{A(r)h(r)\ln^{1/2+\delta}\{h(r)\mu_f(r)\}}\le$$

$$\le\frac{h(r)\mu_f(r)\ln^{1/2+\delta}h(r)\ln^{1/2+\delta}\{h(r)\mu_f(r)\}}{h(r)\ln^{1/2+\delta}\{h(r)\mu_f(r)\}}=\mu_f(r)\ln^{1/2+\delta}h(r)\tag{9}$$

for $r\notin E$, where E is the set of finite h-measure.

We put $k(r)=h(r)\mu_f(r)$ in Lemma 2.2 and let $(r_k)_{k\ge0}$ be the sequence for which consequences of this lemma are valid. We denote by F_k the set of $t\in\mathbb{R}$ such that

$$W(r_k)=\max_{0\le\psi\le2\pi}\Big|\sum_{n\le[C_1(r_k)]}a_nr_k^ne^{in\psi}e^{i\theta_nt}\Big|\ge A_{\beta q}S_{[C_1(r_k)]}(r_k)\ln^{1/2}[C_1(r_k)].$$

It follows from Lemma 2.1 with $\beta=2$ that

$$\sum_{k=1}^{+\infty}P(F_k)\le\sum_{k=1}^{+\infty}\frac{1}{[C_1(r_k)]^2}\le\sum_{k=1}^{+\infty}\frac{1}{[\ln\{\mu_f(r_k)h(r_k)\}]^2}\le\sum_{k=1}^{+\infty}\frac{4}{k^2}<+\infty.$$

Then by Borel-Cantelli's lemma for $k\ge k_0(t)$ and almost surely for $t\in\mathbb{R}$ we obtain

$$W(r_k)<A_qS_{[C_1(r_k)]}(r_k)\ln^{1/2}[C_1(r_k)].\tag{10}$$

From inequalities (6), (10) and $S_{[C_1(r)]}(r) \le M_f(r)\mu_f(r)$ it follows that

$$W(r_k) < \sqrt{\mu_f(r_k)}\sqrt{\mu_f(r_k)h(r_k)}\ln^{1/4}\{h(r_k)\mu_f(r_k)\} \times$$
$$\times \ln^{1/4+2\delta/3} h(r_k)\ln_2^{1/2+2\delta/3}\{h(r_k)\mu_f(r_k)\}\ln^{1/2}(h^2(r)\ln^2\{h(r)\mu_f(r)\}) \le$$
$$\le \mu_f(r_k)\sqrt{h(r_k)}\ln^{1/4}\{h(r_k)\mu_f(r_k)\}\ln^{3/4+3\delta/4} h(r_k)\ln_2^{1+3\delta/4}\{h(r_k)\mu_f(r_k)\}. \tag{11}$$

Since $M_f(r,t) \le \sum_{n\ge C_1(r)} |a_n|r^n + W(r)$, from (9) and (11) we get

$$M_f(r_k,f) \le \mu_f(r_k)\sqrt{h(r_k)}\ln^{1/4}\{h(r_k)\mu_f(r_k)\} \times$$
$$\times \ln^{3/4+4\delta/5} h(r_k)\ln_2^{1+4\delta/5}\{h(r_k)\mu_f(r_k)\}. \tag{12}$$

We suppose that $r_{k_2(t)} \in (0,1)$ is some number outside the set E. Then for $r \in (r_p, r_{p+1})$, $p > k_2(t)$ by Lemma 2.2 we obtain

$$\mu_f(r_{p+1})h(r_{p+1}) \le e\mu_f(r_p)h(r_p) \le e\mu_f(r)h(r), \tag{13}$$

$$\mu_f(r_{p+1}) = h(r_{p+1})\frac{\mu_f(r_{p+1})}{h(r_{p+1})} \le eh(r_p)\frac{\mu_f(r_p)}{h(r_{p+1})} \le eh(r)\frac{\mu_f(r)}{h(r_{p+1})} \le e\mu_f(r), \tag{14}$$

$$h(r_{p+1}) = \frac{\mu_f(r_{p+1})h(r_{p+1})}{\mu_f(r_{p+1})} \le e\frac{\mu_f(r)h(r)}{\mu_f(r_{p+1})} \le eh(r). \tag{15}$$

Finally, from (12) we have for $r \in (r_p, r_{p+1})$

$$M_f(r,t) \le M_f(r_{p+1},t) \le$$
$$\le \mu_f(r)\sqrt{h(r)}\ln^{1/4}\{h(r)\mu_f(r)\}\ln^{3/4+\delta} h(r)\ln_2^{1+\delta}\{h(r)\mu_f(r)\}$$

almost surely for $t \in \mathbb{R}$. The Theorem 3.1 is proved.

By $\mathcal{L}$ we denote the class of increasing to $+\infty$ functions $l(x)$ on $[0,+\infty)$. Let

$$\gamma(l) = \overline{\lim}_{x\to+\infty}\frac{\ln l(x)}{\ln x}.$$

Now we consider the class of analytic functions of the form (3), for which the sequence $(\theta_n)_{n\ge0}$ satisfies the condition

$$\frac{\theta_{n+1}}{\theta_n} \ge 1 + \frac{1}{\varphi(n)},\ \varphi \in \mathcal{L}. \tag{16}$$

What constant instead of $1/4$ *can we put in the inequality* (8) *for this class of analytic functions? Under which conditions on the function* $\varphi(x)$ *does the inequality* (8) *hold?* We give answer to these questions in Corollaries 3.1 and 3.2.

Firstly, we note that in order to get sharper inequality than (2) function $\varphi(x)$ cannot increase rapidly. Indeed, if $\varphi(x) = x$, then we may choose

$$\theta_n = n,\ h(r) = (1-r)^{-1} \quad \text{and} \quad g(z) = \sum_{n=0}^{+\infty} e^{\sqrt{n}}z^n.$$

As it is known from [3],

$$M_g(r,t)=\max\{|g(r,t)|\colon |z|\le r\}=\max_{0\le\psi\le 2\pi}\Big|\sum_{n=0}^{+\infty}a_nr^ne^{int}e^{in\psi}\Big|=$$
$$=\max_{0\le\psi\le 2\pi}\Big|\sum_{n=0}^{+\infty}a_nr^ne^{in(t+\psi)}\Big|=\max_{0\le\psi\le 2\pi}\Big|\sum_{n=0}^{+\infty}a_nr^ne^{in\psi}\Big|=$$
$$=M_g(r)\ge C_1\mu_g(r)h(r)\ln^{1/2}\{\mu_g(r)h(r)\},$$

when $r\to 1-0$ and $t\in\mathbb{R}$. So, in order to improve inequality (2) $\varphi(x)$ must satisfy the condition $\gamma(\varphi)<1$.

Theorem 3.2. *Let $f_t(z)$ be an analytic function of the form* (3)*, $h\in H$, sequence $(\theta_n)_{n\ge0}$ satisfy condition* (16)*, where $\varphi\in\mathcal{L}$. If $v\in\mathcal{L}$ and $\gamma(v)\le 1/4$, then almost surely for $t\in\mathbb{R}$, all $\varepsilon>0$ there exists a set $E(\varepsilon,t)\subset(0,1)$ such that h-meas$(E(\varepsilon,t))<+\infty$ and for $r\in(0,1)\backslash E(\varepsilon,t)$ we have*

$$M_f(r,t)\le\sqrt{h(r)\ln h(r)}\mu_f(r)\ln^{1/4}\{h(r)\mu_f(r)\}\ln^{1+\varepsilon}\{\ln h(r)\ln\{h(r)\mu_f(r)\}\}\times$$
(17)
$$\times\Big(v\Big(8h^2(r)\ln\{h(r)\mu_f(r)\}\Big)+\varphi^{\frac12}\Big(\frac{h^{\frac32}(r)\ln^{\frac54}\{h(r)\mu_f(r)\}\ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}}{v(h(r)\ln\{h(r)\mu_f(r)\})}\Big)\Big).$$

In order to prove this theorem we need a lemma from [6].

Lemma 3.5. [6] *If $(\theta_n)_{n\ge0}$ satisfies condition* (16)*, then for all $\beta>0$*

$$P\Big(\max_{0\le\psi\le 2\pi}\Big|\sum\nolimits_{k=1}^{N}a_ke^{ik\psi}e^{i\theta_kt}\Big|\ge A_\beta\Big\{\varphi(N)S_N\ln N\Big\}^{1/2}\Big)\le N^{-\beta},$$

where A_β is a constant which depends only on β.

Proof of Theorem 3.2. By Lemma 2.4 we obtain outside the set of finite h-measure

$$A(r)\le h(r)\ln\{h(r)\mu_f(r)\}\ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}. \tag{18}$$

We put $C(r)=A(r)T(r)$, where

$$T(r)=\frac{\sqrt{h(r)}\ln^{1/4}\{h(r)\mu_f(r)\}}{v(h^2(r)\ln\{h(r)\mu_f(r)\})}.$$

Then from (18) we have

$$C(r)=A(r)T(r)\le h(r)\ln\{h(r)\mu_f(r)\}\ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}T(r)=$$
$$=\frac{h^{3/2}(r)\ln^{5/4}\{h(r)\mu_f(r)\}\ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}}{v(h^2(r)\ln\{h(r)\mu_f(r)\})}=C_1(r).$$

Now using Markov's inequality we get

$$\sum_{n\geq C_1(r)}|a_n|r^n\leq\sum_{n\geq C(r)}|a_n|r^n\leq\frac{G_f(r)}{T(r)}\leq$$
$$\leq\frac{h(r)\mu_f(r)\{\ln h(r)\ln\{h(r)\mu_f(r)\}\}^{1/2}\ln^{1+\delta}\{\ln h(r)\ln\{h(r)\mu_f(r)\}\}}{\sqrt{h(r)}\ln^{1/4}\{h(r)\mu_f(r)\}}\times$$
$$\times v(h^2(r)\ln\{h(r)\mu_f(r)\})=$$
$$=\mu_f(r)\sqrt{h(r)\ln h(r)}\ln^{1/4}\{h(r)\mu_f(r)\}\ln^{1+\delta}\{\ln h(r)\ln\{h(r)\mu_f(r)\}\}\times$$
$$\times v(h^2(r)\ln\{h(r)\mu_f(r)\}). \tag{19}$$

Let

$$k(r)=h(r)\mu_f(r)\quad\text{and}\quad(r_k)_{k\geq0}$$

be the sequence for which consequences of Lemma 2.2 are valid. Denote by G_k the set of such $t\in\mathbb{R}$, for which

$$W_1(r_k)=\max_{0\leq\psi\leq2\pi}\Big|\sum_{n\leq[C_1(r_k)]}a_nr_k^ne^{in\psi}e^{i\theta_nt}\Big|\geq$$
$$\geq A_\beta\Big(\varphi([C_1(r_k)])S_{[C_1(r_k)]}(r_k)\ln[C_1(r_k)]\Big)^{1/2},$$

where

$$S_f^2(r)=\sum_{n=0}^{+\infty}|a_n|^2r^{2n}.$$

Since $\gamma(v)\leq1/4$, we have

$$C_1(r)>\frac{h^{3/2}(r)\ln^{5/4}\{h(r)\mu_f(r)\}\ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}}{(h^2(r)\ln\{h(r)\mu_f(r)\})^{1/4}}>$$
$$>h(r)\ln\{h(r)\mu_f(r)\}\ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}>\ln\{h(r)\mu_f(r)\}.$$

So, by Lemma 2.2 $\ln k(r_n)>n/2$, i.e. $\ln\{h(r_n)\mu_f(r_n)\}>n/2$. Then

$$C_1(r_n)>\ln\{h(r_n)\mu_f(r_n)\}>n/2.$$

Using Lemma 3.1 with $\beta=2$ we get

$$\sum_{k=1}^{+\infty}P(G_k)<\sum_{k=1}^{+\infty}\frac{1}{N^\beta(r_k)}<\sum_{k=1}^{+\infty}\frac{4}{k^2}<+\infty.$$

Now by Borel-Cantelli's lemma for $k\geq k_2(t)$ and almost surely $t\in\mathbb{R}$ we obtain

$$W_1(r_k)<A_\beta\Big(\varphi([C_1(r_k)])S_{[C_1(r_k)]}(r_k)\ln[C_1(r_k)]\Big)^{1/2}.$$

Using the inequality $S_f^2(r) \le G_f(r)\mu_f(r)$, we obtain

$$W_1(r_k) < \sqrt{h(r_k)\ln h(r_k)}\mu_f(r_k)\ln^{1/4}\{h(r_k)\mu_f(r_k)\} \times \\ \times \ln^{1+\delta}\{\ln h(r_k)\ln\{h(r_k)\mu_f(r_k)\}\} \times \\ \times \varphi^{1/2}\Big(\frac{h^{3/2}(r_k)\ln^{5/4}\{h(r_k)\mu_f(r_k)\}\ln_2^{1+\delta}\{h(r_k)\mu_f(r_k)\}}{v(h^2(r_k)\ln\{h(r_k)\mu_f(r_k)\})}\Big). \tag{20}$$

It follows from (20) and (17) that

$$M_f(r_k,t) \le \sqrt{h(r_k)\ln h(r_k)}\mu_f(r_k)\ln^{1/4}\{h(r_k)\mu_f(r_k)\} \times \\ \times \ln^{1+\delta}\{\ln h(r_k)\ln\{h(r_k)\mu_f(r_k)\}\}\Big(v(h^2(r_k)\ln\{h(r_k)\mu_f(r_k)\}) + \\ +\varphi^{1/2}\Big(\frac{h^{3/2}(r_k)\ln^{5/4}\{h(r_k)\mu_f(r_k)\}\ln_2^{1+\delta}\{h(r_k)\mu_f(r_k)\}}{v(h^2(r_k)\ln\{h(r_k)\mu_f(r_k)\})}\Big)\Big).$$

Using (13)–(15) we get for $r \in (r_p, r_{p+1})$

$$M_f(r,t) \le \sqrt{h(r)\ln h(r)}\mu_f(r)\ln^{1/4}\{h(r)\mu_f(r)\} \times \\ \times \ln^{1+2\delta}\{\ln h(r)\ln\{h(r)\mu_f(r)\}\}\Big(v(8h^2(r)\ln\{h(r)\mu_f(r)\}) + \\ +\varphi^{1/2}\Big(\frac{h^{3/2}(r)\ln^{5/4}\{h(r)\mu_f(r)\}\ln^{1+2\delta}\ln\{h(r)\mu_f(r)\}}{v(h^2(r)\ln\{h(r)\mu_f(r)\})}\Big)\Big).$$

Theorem 3.2 is proved.

In the case when

$$\ln\varphi(x) = o(\ln_2 x), \quad x \to +\infty$$

we have the following corollary.

Corollary 3.1. *Let $f_t(z)$ be an analytic function of the form* (3)*, $h \in H$, a sequence $(\theta_n)_{n\ge 0}$ satisfy condition* (16)*, where $\varphi \in \mathcal{L}$ and $\ln\varphi(x) = O(\ln_2 x)$, $x \to +\infty$. Then there exists a set $E(\delta,t) \subset (0,1)$ such that h-meas$(E(\delta,t)) < +\infty$ and almost surely for $t \in \mathbb{R}$ we get*

$$\varlimsup_{\substack{r\to 1-0\\ r\notin E}} \Delta_h(r,f_t) \le \frac{1}{4}.$$

Corollary 3.2. *Let $f_t(z)$ be an analytic function of the form* (3)*, $h \in H$, sequence $(\theta_n)_{n\ge 0}$ satisfy condition* (16)*, where $\varphi \in \mathcal{L}$ and*

$$\gamma(\varphi) = \varlimsup_{n\to+\infty} \frac{1}{\ln n}\ln\frac{\theta_n}{\theta_{n+1}-\theta_n} \le \delta \in [0,1/2). \tag{21}$$

Then for all analytic functions f_t there exists a set $E(\delta,t) \subset (0,1)$ such that h-meas $(E(\delta,t)) < +\infty$ and almost surely for $t \in \mathbb{R}$ we have

$$\varlimsup_{\substack{r\to 1-0\\ r\notin E}} \Delta_h(r,f_t) \le \frac{1+3\delta}{4+2\delta}.$$

Proof. If $\gamma(\varphi)=\delta\in[0,1/2)$, then we may choose $v(x)=x^{\alpha},\ \alpha\in[0,1/4)$. So,

$$\ln v(8h^2(r)\ln\{h(r)\mu_f(r)\})=(\alpha+o(1))(2\ln h(r)+\ln_2\{h(r)\mu_f(r)\}),$$

$$\ln\Big(\varphi^{1/2}\Big(\frac{h^{3/2}(r)\ln^{5/4}\{h(r)\mu_f(r)\}\ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}}{v(h^2(r)\ln\{h(r)\mu_f(r)\})}\Big)\Big)\le$$

$$\le(1+o(1))\Big(\frac{3\delta}{4}\ln h(r)+\frac{5\delta}{8}\ln_2\{h(r)\mu_f(r)\}-\delta\alpha\ln h(r)-\frac{\delta\alpha}{2}\ln_2\{h(r)\mu_f(r)\}\Big)=$$

$$=\Big(\frac{3\delta}{8}-\frac{\delta\alpha}{2}+o(1)\Big)2\ln h(r)+\Big(\frac{5\delta}{8}-\frac{\delta\alpha}{2}+o(1)\Big)\ln_2\{h(r)\mu_f(r)\}\le$$

$$\le\Big(\frac{5\delta}{8}-\frac{\delta\alpha}{2}+o(1)\Big)(2\ln h(r)+\ln_2\{h(r)\mu_f(r)\}). \tag{22}$$

From the equation $\alpha=\frac{5\delta}{8}-\frac{\delta\alpha}{2}$ we may choose $\alpha=\frac{5\delta}{4(2+\delta)}$ and get as $r\to1-0$

$$\ln M_f(r,t)\le(1+o(1))\Big(\frac{1}{2}\ln h(r)+\ln\mu_f(r)+$$
$$+\frac{1}{4}\ln_2\{h(r)\mu_f(r)\}+\alpha(\ln h(r)+\ln_2\{h(r)\mu_f(r)\})\Big).$$

Therefore,

$$\overline{\lim}_{\substack{r\to1-0\\ r\notin E}}\Delta_h(r,f_t)=\overline{\lim}_{\substack{r\to1-0\\ r\notin E}}\frac{\ln M_f(r,t)-\ln\mu_f(r)}{2\ln h(r)+\ln_2\{h(r)\mu_f(r)\}}\le$$
$$\le\frac{1}{4}+\alpha=\frac{1}{4}+\frac{5\delta}{4(2+\delta)}=\frac{1+3\delta}{4+2\delta}.$$

Corollary 3.2 is proved.

So, we can improve inequality (2) for all analytic functions of the form (3) and all $h\in H$, when $\gamma(\varphi)<1/2$. As it is remarked above, this inequality cannot be improved, if $\gamma(\varphi)\ge1$. Can we improve inequality (2) for all analytic functions of the form (3) by condition $\gamma(\varphi)<1$?

Corollary 3.3 gives a positive answer to this question by some choice of the function $h(r)$.

Corollary 3.3. *Let $f_t(z)$ be an analytic function of the form* (3)*,*

$$h\in H\colon\quad \ln_2\mu_f(r)=o(\ln h(r)),\quad r\to1-0,$$

a sequence $(\theta_n)_{n\ge0}$ satisfy condition (16)*, where $\varphi\in\mathcal{L}$ and*

$$\gamma(\varphi)=\overline{\lim}_{n\to+\infty}\frac{1}{\ln n}\ln\frac{\theta_n}{\theta_{n+1}-\theta_n}\le\delta\in[0,1). \tag{23}$$

Then for all analytic functions f_t there exists a set $E(\delta,t)\subset(0,1)$ such that h-meas $(E(\delta,t))<+\infty$ and almost surely for $t\in\mathbb{R}$

$$\overline{\lim}_{\substack{r\to1-0\\ r\notin E}}\Delta_h(r,f_t)\le\frac{1+2\delta}{4+2\delta}.$$

Proof. It follows from (22) that

$$\ln\Big(\varphi^{1/2}\Big(\frac{h^{3/2}(r)\ln^{5/4}\{h(r)\mu_f(r)\}\ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}}{v(h^2(r)\ln\{h(r)\mu_f(r)\})}\Big)\Big)\leq$$
$$\leq\Big(\frac{3\delta}{8}-\frac{\delta\alpha}{2}+o(1)\Big)2\ln h(r)+\Big(\frac{5\delta}{8}-\frac{\delta\alpha}{2}+o(1)\Big)\ln_2\{h(r)\mu_f(r)\}\leq$$
$$\leq\Big(\frac{3\delta}{8}-\frac{\delta\alpha}{2}+o(1)\Big)2\ln h(r)\leq\Big(\frac{3\delta}{8}-\frac{\delta\alpha}{2}+o(1)\Big)(2\ln h(r)+\ln_2\{h(r)\mu_f(r)\}).$$

From equation

$$\alpha=\frac{3\delta}{8}-\frac{\delta\alpha}{2}\quad\text{we choose}\quad\alpha=\frac{3\delta}{4(2+\delta)}$$

and

$$\varlimsup_{\substack{r\to1-0\\ r\notin E}}\Delta_h(r,f_t)=\varlimsup_{\substack{r\to1-0\\ r\notin E}}\frac{\ln M_f(r,t)-\ln\mu_f(r)}{2\ln h(r)+\ln_2\{h(r)\mu_f(r)\}}\leq$$
$$\leq\frac{1}{4}+\alpha=\frac{1}{4}+\frac{3\delta}{4(2+\delta)}=\frac{1+2\delta}{4+2\delta}.$$

Corollary 3.3 is proved.

4. Baire's categories and Wiman-Valiron's type inequality for analytic functions

Let $h\in H$ and $\theta=(\theta_n)_{n\geq0}$ be a fixed sequence satisfying condition (21), such that $\gamma(\varphi)\leq\delta$. Similarly to [11], we define the following sets

$$F_{1h}(f,\theta,E)=\Big\{t\in\mathbb{R}\colon\ \varlimsup_{\substack{r\to1-0\\ r\notin E}}\Delta_h(r,f_t)\leq\frac{1+3\delta}{4+2\delta}\Big\}$$

$$F_{2h}(f,\theta)=\Big\{t\in\mathbb{R}\colon\ \Big(\forall\,\eta>\frac{1+3\delta}{4+2\delta}\Big)\big[h\text{-meas}(E(\eta,f_t,h))<+\infty\big]\Big\}$$

$$F_{3h}(f,\theta)=\Big\{t\in\mathbb{R}\colon\ \varliminf_{r\to1-0}\Delta_h(r,f_t)\leq\frac{1+3\delta}{4+2\delta}\Big\},$$

$$F_{4h}(f,\theta)=\Big\{t\in\mathbb{R}\colon\ \varliminf_{r\to1-0}\Delta_h(r,f_t)\leq\frac{1+2\delta}{4+2\delta}\Big\}.$$

By Corollary 3.2 we conclude that for analytic functions in $\mathbb{D}$ there exists the set $E(f)$ of finite h-measure such that the set $F_{1h}(f,\theta)$ is "large" in the sense of Lebesque measure. Therefore, we obtain some information on sets $F_{2h}(f,\theta)$, $F_{3h}(f,\theta)$.

Similarly to [11], the following question arises naturally: *does there exists a set $E=E(f)$ of the finite h-measure such that the set $F_{1h}(f,\theta,E)$ is residual in $\mathbb{R}$ for every analytic function f?*

We recall that a set $B\subset\mathbb{R}$ is called residual in $\mathbb{R}$, if its complement $\overline{B}=\mathbb{R}\backslash B$ is a set of the first Baire category in $\mathbb{R}$. It is clear, that if the answer to the question is affirmative, then the sets $F_{2h}(f,\theta)$, $F_{3h}(f,\theta)$ are residual in $\mathbb{R}$. However for some analytic function the set $F_{1h}(f,\theta,E)$ is a set of the first Baire category (see similar assertion for the entire function $f(z)=e^z$ in [11]). It follows from the following theorem.

Theorem 4.3. *Let a sequence $(\theta_n)_{n\geq 0}$ satisfy condition (4),*

$$f(z)=\sum_{n=0}^{+\infty} e^{n^{\varepsilon}} z^n, \quad \varepsilon\in(0,1), \quad \textit{and} \quad h(r)=(1-r)^{-1}.$$

Then there exists a constant $C=C(\theta,\varepsilon)>0$ such that for all sequences $(r_n)_{n\geq 0}$ increasing to 1 the set

$$F_3=\Big\{t\in\mathbb{R}\colon\ \overline{\lim}_{n\to+\infty}\frac{M_{f_t}(r_n)}{h(r_n)\mu_f(r_n)\ln^{1/2}\{h(r_n)\mu_f(r_n)\}}\leq C\Big\}$$

is a set of the first Baire category.

Lemma 4.6. [12] *For every $q>1$ there exist positive constants $A=A(q)$ and $B=B(q)$ such that for each interval $I\subset\mathbb{R}$ and every trigonometrical polynomial*

$$Q(t)=\sum_{n=1}^{N} c_n e^{i\lambda_n t}, \quad 0<\lambda_1<\lambda_2<\ldots<\lambda_N,$$

for which

$$|I|\geq\frac{B}{\lambda_1}>0 \quad \textit{and} \quad \frac{\lambda_{n+1}}{\lambda_n}\geq q, \quad 1\leq n\leq N-1,$$

there exists a point $t_0\in I$ such that

$$\operatorname{Re}Q(t_0)\geq A\sum\nolimits_{n=1}^{N}|c_n|.$$

Proof of Theorem 4.1. For the function

$$f(z)=\sum_{n=1}^{+\infty}\exp\{n^{\varepsilon}\}z^n, \quad \varepsilon\in(0,1)$$

(see [3]) there exists $C_0(\varepsilon)\in(0,1)$ such that we have

$$C_0^{-1}(\varepsilon)\frac{\mu_f(r)}{1-r}\geq\frac{M_f(r)}{\sqrt{\ln M_f(r)}}\geq C_0(\varepsilon)\frac{\mu_f(r)}{1-r},\ r\to 1-0.$$

Then we obtain as $r\to 1-0$

$$\ln M_f(r)-\frac{1}{2}\ln_2 M_f(r)\geq\ln C_0(\varepsilon)+\ln\frac{\mu_f(r)}{1-r},\ \ln M_f(r)\geq\ln\frac{\mu_f(r)}{1-r},$$

$$M_f(r)\geq C_0(\varepsilon)\frac{\mu_f(r)}{1-r}\ln^{1/2}\frac{\mu_f(r)}{1-r}. \tag{24}$$

Let $(r_n)_{n\geq 0}$ be sequence some increasing to 1. We put $A = A(q)$ and $B = B(q)$ are the constants from Lemma 4.1,

$$C(\varepsilon) = AC_0(\varepsilon), \quad q = \inf\{\frac{\theta_{n+1}}{\theta_n} : n \geq 0\}.$$

We consider the sequence of the numbers $(C_n(\varepsilon))_{n\geq 0}$ increasing to $C(\varepsilon)$. Define the set

$$F_{mk} = \big\{t \in \mathbb{R}\colon \ (\forall l \geq k)\big[M_{f_t}(r_l) \leq C_m(\varepsilon)\frac{\mu_f(r_l)}{1-r_l}\ln^{1/2}\frac{\mu_f(r_l)}{1-r_l}\big]\big\}.$$

where integers $k \geq 0, \ m \geq 0$ are fixed. For fixed $r \in (0,1)$ we consider the function

$$\alpha(t,\varphi) = \Big|\sum\nolimits_{n=0}^{+\infty} \exp\{i\theta_n t + n^{\varepsilon} + in\varphi\}r^n\Big|,$$

which is continuous in $\mathbb{R}^2$ and periodic in the variables t and φ. Then the function

$$\beta(t) = \max_{\varphi} \alpha(t,\varphi) = M_f(r,t)$$

continuous at every point $t \in \mathbb{R}$. We remark, that the set F_{mk} is closed in $\mathbb{R}$.

Now we prove that the set $\overline{F_{mk}}$ is everywhere dense. Consider an arbitrary interval $I \subset \mathbb{R}, \ |I| > 0$ and show that it contains some point t_0 from the set $\overline{F_{mk}}$.

Let us choose $p \geq 1, \delta > 0$ such that

$$|I| \geq \frac{B}{\theta_p}, \ 1 - 2\delta > \sqrt{\frac{C_m(\varepsilon)}{C(\varepsilon)}}. \tag{25}$$

Using (24), we may define

$$x_1 = x_1(\varepsilon) = \inf\Big\{r \in (0,1)\colon \sum_{n=0}^{+\infty} \exp\{n^{\varepsilon}\}r^n \geq (1-2\delta)C_0(\varepsilon)\frac{\mu_f(r)}{1-r}\ln^{1/2}\frac{\mu_f(r)}{1-r}\Big\},$$

$$x_2 = x_2(\varepsilon) = \inf\Big\{r \in (0,1)\colon \sum_{n=0}^{p} \exp\{n^{\varepsilon}\}r^n \leq \frac{A}{A+1}\delta\sum_{n=0}^{+\infty}\exp\{n^{\varepsilon}\}r^n\Big\}.$$

Now choose the integers $l \geq k$ and $s > p$ such that inequalities

$$r_l > \max\{x_1, x_2\}, \ \sum_{n=s+1}^{+\infty} \exp\{n^{\varepsilon}\}r_l^n \leq \frac{A}{A+1}\delta\sum_{n=0}^{+\infty}\exp\{n^{\varepsilon}\}r_l^n \tag{26}$$

hold. By Lemma 2.2 there exists a point t_0 in the interval I such that

$$\mathrm{Re}\Big(\sum_{n=p}^{s} e^{i\theta_n t_0 + n^{\varepsilon}} r_l^n\Big) \geq A\sum_{n=p}^{s}\exp\{n^{\varepsilon}\}r_l^n. \tag{27}$$

Using the definitions of x_1, x_2 from (24)–(27) we deduce

$$M_f(r_l, t_0) = \max_{\varphi} |f_{t_0}(r_l e^{i\varphi})| \geq$$

$$\geq |f_{t_0}(r_l)| \geq \operatorname{Re} f_{t_0}(r_l) \geq \operatorname{Re}\Big(\sum_{n=p}^{s} \exp\{i\theta_n t_0 + n^{\varepsilon}\} r_l^n\Big) -$$

$$- \sum_{n \notin [p,s]} \exp\{n^{\varepsilon}\} r_l^n \geq A \sum_{n=p}^{s} \exp\{n^{\varepsilon}\} r_l^n - \sum_{n \notin [p,s]} \exp\{n^{\varepsilon}\} r_l^n =$$

$$= A \sum_{n=0}^{+\infty} \exp\{n^{\varepsilon}\} r_l^n - (1 + A) \sum_{n \notin [p,s]} \exp\{n^{\varepsilon}\} r_l^n \geq$$

$$\geq A \sum_{n=0}^{+\infty} \exp\{n^{\varepsilon}\} r_l^n - (1 + A) \frac{2A}{1 + A} \delta \sum_{n=0}^{+\infty} \exp\{n^{\varepsilon}\} r_l^n =$$

$$= A(1 - 2\delta) \sum_{n=0}^{+\infty} \exp\{n^{\varepsilon}\} r_l^n \geq \frac{C(\varepsilon)}{C_0(\varepsilon)} (1 - 2\delta)^2 C_0(\varepsilon) \frac{\mu_f(r_l)}{1 - r_l} \ln^{1/2} \frac{\mu_f(r_l)}{1 - r_l} \geq$$

$$\geq C_m(\varepsilon) \frac{\mu_f(r_l)}{1 - r_l} \ln^{1/2} \frac{\mu_f(r_l)}{1 - r_l}.$$

Therefore, $t_0 \in \overline{F_{mk}}$. Since the set F_{mk} is closed in $\mathbb{R}$ and its complement $\overline{F_{mk}}$ is everywhere dense, the set F_{mk} is nowhere dense. Hence

$$F_3 = \bigcup_{m=0}^{+\infty} \bigcup_{k=0}^{+\infty} F_{mk}$$

is a set of the first Baire category. Theorem 4.1 is proved.

Theorem 4.2. *If sequence $(\theta_n)_{n \geq 0}$ satisfies condition* (21) *and $h \in H$, then for every analytic function f the set $F_{3h}(f, \theta)$ is residual in $\mathbb{R}$.*

Proof. Let f be an arbitrary analytic function in $\mathbb{D}$. We consider the sequence $(c_n)_{n \geq 0}$ such that

$$c_n \downarrow \frac{1 + 3\delta}{4 + 2\delta}, \ n \to +\infty.$$

Fix integers $m \geq 0, \ k \geq 0$ and define the set

$$G_{mk} = \Big\{ t \in \mathbb{R} \colon M_f(r, t) \geq \frac{\mu_f(r)}{(1 - r)^{c_m}} \ln^{c_m} \frac{\mu_f(r)}{1 - r}, \ \forall r > 1 - \frac{1}{k + 1} \Big\}.$$

As it has been proved above, for every fixed $r \in (0, 1)$ function $\beta(t) = M_f(r, t)$ is continuous at every point $t_0 \in \mathbb{R}$. Then the set G_{mk} is closed in $\mathbb{R}$. By Corollary 3.2 the set $\overline{G_{mk}}$ is everywhere dense. Therefore, G_{mk} is nowhere dense and

$$G = \bigcup_{m=0}^{+\infty} \bigcup_{k=0}^{+\infty} G_{mk}$$

is a set of the first Baire category. So, $F_{3h}(f, \theta) = \overline{G}$ residual set in $\mathbb{R}$. Theorem 4.2 is proved.

Theorem 4.3. *If sequence* $(\theta_n)_{n\geq 0}$ *satisfies condition* (23) *and* $h \in H$*, then for all analytic functions* f *such that* $\ln_2 \mu_f(r) = o(\ln h(r))$, $r \to 1-0$*, the set* $F_{4h}(f,\theta)$ *is residual in* $\mathbb{R}$.

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KATEGORIE BAIRE'A I NIERÓWNOŚĆ WIMANA DLA FUNKCJI ANALITYCZNYCH

S t r e s z c z e n i e

Niech $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ $(z \in \mathbb{C})$ będzie funkcją analityczną w kole jednostkowym i niech f_t będzie funkcją analityczną postaci $f_t = \sum_{n=0}^{+\infty} a_n e^{i\theta_n t} z^n$, gdzie $t \in \mathbb{R}$, $\theta_n \in \mathbb{N}$, h zaś jest dodatnią funkcją ciągłą na odcinku $(0,1)$, rosnącą do $+\infty$ i taką, że $\int_0^1 h(r)dr = +\infty$. Dowodzimy, że jeśli ciąg $(\theta_n)_{n\geq 0}$ spełnia nierówność (*), to dla każdej fukcji analitycznej f dla prawie każdego t istnieje taki zbiór $E = E(\delta, t) \subset (0,1)$, że $\int_E h(r)dr < +\infty$ oraz zachodzi oszacowanie (**), gdzie $M_f(r,t) = \max\{|f_t(z)|\colon |z| = r\}$, $\mu_f(r) = \max\{|a_n|r^n\colon n \geq 0\}$ dla $r \in [0,1)$.

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In memory of
Professor Promarz M. Tamrazov

Zhidong Zhang and Norman H. March

CONFORMAL INVARIANCE IN THE THREE-DIMENSIONAL (3D) ISING MODEL AND QUATERNIONIC GEOMETRIC PHASE IN QUATERNIONIC HILBERT SPACE

Summary

Based on the quaternionic approach developed by one of us [Z.-D. Zhang, Phil. Mag. **88** (2008), 3097.] for the three-dimensional (3D) Ising model, we study in this work conformal invariance in three dimensions. We develop a procedure for treating the 3D conformal field theory. The 2D conformal field theory is extended to be appropriate for three dimensions, within the framework of quaternionic coordinates with complex weights. The Virasoro algebra still works, but for each complex plane of quaternionic coordinates. The quaternionic geometric phases appear in quaternionic Hilbert space as a result of diagonalization procedure which involves the smoothing of knots/crossings in the 3D many-body interacting spin Ising system. Possibility for application of conformal invariance in three dimensions on studying the behaviour of the world volume of the brane, or the world sheet of the string in 3D or (3+1)D, is briefly discussed.

Keywords and phrases: Ising lattice, quaternionic phase, quaternionic Hilbert space, conformal invariance in 3 dimensions

1.

It is well known that in the context of a physical system with local interactions, conformal invariance is an immediate extension of scale invariance, a symmetry under dilations of space [1]. Conformal transformations are dilations by a scaling factor that is a function of position (local dilations) [2]. The study of conformal invariance in two dimensions was initiated by Belavin, Polyakov and Zamolodchikov, which combined the representation theory of the Virasoro algebra with the idea of an algebra of local operators [3]. Conformal field theories not only provide toy models for genuinely interacting quantum field theory, but also play a central role in string theory [4] where two-dimensional (2D) scale invariance appears naturally. The vibrations of a string with internal degrees of freedom can be studied within a conformal field theory, from the point view of the world sheet. A classification of the 2D conformal field would provide useful information on the classical solution space of string theory [5]. Conformal field theories also describe critical phenomena at the critical point of the second-order phase transitions in two dimensions, where the correlation length diverges [5]. The partition function of a statistical mechanics model can be closely related to the knot polynomials by matrix elements of the braiding matrices of an associated rational conformal field theory, or alternatively, the matrix elements of the R-matrix of a quantum group [6, 7].

The critical point of the 2D Ising model, as a canonical example, is described by a conformal field theory, since every scale-invariant 2D local quantum field theory is actually conformal invariant. The high-temperature disordered phase and the low-temperature ordered phase in the 2D Ising model are related by a duality of the model, and the second order phase transition occurs at the self-dual point [8].

A finite number of parameters ($(d+1)(d+2)/2$) are needed to specify a conformal transformation in d spatial dimensions [1]. The consequence of this finiteness is that in three or more dimensions, conformal invariance does not turn out to give much more information than ordinary scale invariance [5]. It is commonly accepted that the exception is in two dimensions, where the number of parameters specifying local conformal transformations is infinite. In two dimensions, the conformal algebra becomes infinite dimensional, leading to significant restrictions on the 2D conformally invariant theories [5]. It means that an infinite variety of conformal transformations exist in 2D, which, although not everywhere well defined, are locally conformal [1]: they are holomorphic mappings from the complex plane (or part of it) onto itself. A local field theory should be sensitive to local symmetries, even if the related transformations are not globally defined. It is local conformal invariance that enables exact solutions of 2D conformal field theories [1]. This is the reason for the success of conformal invariance in the study of 2D critical systems.

2.

One of us (ZDZ) [9] has worked on the three-dimensional (3D) Ising model, using a quaternionic approach, which has recently found favour with mathematicians [10–12]. It was pointed out in Ref. [9, 13, 14] that the framework of the statistical mechanics for 3D Ising magnets should include the time, being in the $(3 + 1)$ dimensional Euclidean spacetime. This argument is based on a fact that the temperature in statistical mechanics is actually the time in quantum field theory [15]. This is because the Euclidean time interval can be consistently identified with β. There are serious challenges to the validation of the ergodic hypothesis in the 3D many-body interacting spin systems, like the 3D Ising model, where the topologic contributions to the partition function as well as correlation functions and other physical quantities cannot be negligible [9, 13, 14]. In a more recent work [16], we represented a detailed analysis of temperature-time duality in the 3D Ising model. It was pointed out that the time necessary for the time averaging must be infinite, being comparable with or even much far than the time of measurement of the physical quantity of interests, because the topologic effects are non-local so that an efficient sweep of the ensemble by any of its microstates should be infinite since the number of these microstates is infinite [16]. Therefore, the integrand of the partition function of the 3D Ising model should be performed in four dimensions, since one needs to take the time average by the integrand in the fourth dimension.

In this work, we develop a procedure for treating the 3D conformal field theory. We study the conformal invariance in the 3D Ising model by the quaternionic approach developed in Ref. [9] and discuss the quaternionic geometric phase in quaternionic Hilbert space. We uncover that for treating the 3D conformal field theory, the representation theory of the Virasoro algebra and the algebra of local operators should be combined with the quaternionic geometric phase in quaternionic Hilbert space. The decomposition of 3D conformal blocks to 2D conformal blocks can be done by the utilization of Jordan-von Neumann-Wigner procedure [17].

The success of studying the 3D Ising model within the $(3+1)$D framework provides a chance to understand deeper conformal invariance and conformal transformations in three dimensions. From another point of view, it is anticipated that the procedure for the exact solution of the 3D Ising model enables local conformal invariance in three dimensions. It is possible to remove the finiteness parameters needed to specify a conformal transformation in three dimensions, so that the conformal algebra becomes infinite dimensional. This implies that, in order to maintain local conformal invariance in three dimensions, the exact solution of the 3D Ising model should possess the character of the exact solution of the 2D Ising model, as what Zhang proposed in Ref. [9]. Indeed, the exact solution of the 3D Ising model, followed by means of two conjectures, has the main feature of the Onsager's exact solution of the 2D Ising model, while the only difference between the solutions in 3D and 2D is the appearance of weight factors in the partition function and eigenvectors within the $(3+1)$D quaternionic framework [9].

3.

The conformal group in d-dimension is the subgroup of coordinate transformations that leaves the metric invariant up to a scale change [1, 3–5],

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{1}$$

Here we consider the space R^d with flat metric $g_{\mu\nu}(x) = \eta_{\mu\nu}(x)$ with signature (p, q) and line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. For $d > 2$, the conformal algebra is isomorphic to SO $(p+1, q+1)$ (Ref. [5]). Actually, the conformal group admits a nice realization acting on $\mathrm{R}^{p,q}$, stereographically projected to $\mathrm{S}^{p,q}$, and embedded in the light cone of $\mathrm{R}^{p+1,q+1}$ (see Ref. [5]).

The 2D conformal field theory is naturally defined on a Riemann surface (or complex curve), i.e., on a surface possessing suitable complex coordinates. In the case of the sphere, the complex coordinates can be taken to be those of the complex plane that cover the sphere except for the point at infinity. The quaternionic approach developed in Ref. [9] provides a possibility to deal with the 3D conformal field theory in the bases of the Hilbert space with suitable quaternionic coordinates. Here we suggest that the Hilbert space of the 3D Ising model, so-called complexified quaternionic Hilbert space, provides the important information of the conformal field theory in three or $(3+1)$ dimensions. So the correlation functions in the 3D Ising systems and also the 3D conformal field theory depend on the quaternionic parameters. The conformal invariance in the 3D Ising model can be studied also with the quaternionic coordinates. In what follows, we will illustrate how to process the conformal transformation in three dimensions.

For a 3D many-body interacting system, like the 3D Ising model or the 3D conformal field theory, we need to decompose 3D conformal blocks to 2D conformal blocks by the utilization of Jordan-von Neumann-Wigner procedure [17], which can be realized by introducing an additional dimension, performing a unitary transformation as a rotation in higher dimensions, and constructing a quaternionic basis together with weight factors [9]. The normalized eigenvectors (eq. (33) in Ref. [9]), proposed for the 3D Ising model, are quaternionic eigenvectors (also see Ref. [10, 11, 12] for the mathematical outlook). The complexified weight factors have significance of topological phases as revealed in Ref. [9, 13, 14]. Actually, the quaternionic bases found for the 3D Ising model in Ref. [9] are complexified quaternionic bases. One refers to Ref. [18] for details of complexified quaternion, Ref. [19, 20, 21] for quaternionic quantum mechanics, and Ref. [22] for quaternion and special relativity. The use of Clifford structures and the P. Jordan structures can make the way of applying the quaternion structure more elegant and simpler [10–12]. Jordan algebras with its multiplication $\mathrm{A} \circ \mathrm{B} = \frac{1}{2}\,(\mathrm{AB}+\mathrm{BA})$ instead of the usual matrix multiplication AB replaces in an elegant way the desire of looking for commutative subalgebras of the algebra constructed and for combinatorial properties. Such desire is one of the main obstacles in solving exactly the 3D Ising model. We believe that it is also the case for the 3D conformal field theory.

4.

For the 3D conformal field theory, therefore, one can define the quaternionic coordinates

$$q = x^0 + iw_1x^1 + jw_2x^2 + kw_3x^3 \quad \text{and} \quad \bar{q} = x^0 - iw_1x^1 - jw_2x^2 - kw_3x^3.$$

Here w_i $(i = 1, 2, 3)$ are complex weight factors on the imaginary coordinates of the quaternionic bases. The 3D conformal transformations can thus be decomposed into three 2D conformal transformations that coincide with the analytic coordinate transformations $z_i \to f_i(z_i)$ and $\bar{z}_i \to f_i(\bar{z}_i)$ $(i = 1, 2, 3)$, the local algebra of each of which is infinite dimensional as what we have for the 2D conformal field theory. In the complexified quaternionic coordinates, we have

$$ds^2 = dq d\bar{q} \to \sum_{i=1}^{3} \left| \frac{\partial f_i}{\partial z_i} \right|^2 |w_i|^2 dz_i d\bar{z}_i. \tag{2}$$

Therefore, the 2D conformal field theory can be generalized to the 3D conformal field theory in such a way: Each imaginary coordinate in the quaternionic coordinates and the real coordinate construct the complex coordinates for the 2D conformal field theory. We can study the conformal invariance within such complex coordinates, keeping in mind that there are complex weight factors w_i $(i = 1, 2, 3)$ for each imaginary coordinate in the quaternionic coordinates. The differences between eq. (2) for the 3D conformal field theory and that for the 2D conformal field theory in literature [1, 3–5] are as follows: 1) the summation w.r.t. i and 2) the phase factors w_i.

The conformal transformation in three dimensions can be written as

$$\tau_i \longrightarrow A_i \tau_i = \frac{a_i \tau_i + b_i}{c_i \tau_i + d_i} \tag{3}$$

with the matrix

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix},$$

where $a_i d_i - b_i c_i = 1$,

Again, $i = 1, 2$ or 3 corresponds to the complex plane constructed by the real coordinate and one of the three imaginary coordinates of quaternionic bases. Although each conformal transformation in eq. (3) for three dimensions has the same feature as the conformal transformation for two dimensions in literature [1,3–5], one should notice that there are three different conformal transformations performed in the quaternionic bases for a 3D system.

5.

Following the procedure for the 2D conformal field theory [1, 3–5], the energy-momentum tensor in the 3D systems (in (3 + 1)D framework) can be expanded as

$$L(z) = \sum_{i=1}^{3} \sum_{n=-\infty}^{\infty} |w_i| \mathrm{Re} |e^{i\phi_i}| L_{n_i} z_i^{-n-2} \tag{4}$$

with the commutator as

$$[L_{m_i}, L_{n_i}] = (m_i - n_i) L_{m_i + n_i} + \frac{c}{12} m_i (m_i^2 - 1) \delta_{m_i + n_i, 0}; \tag{5}$$

here $i = 1, 2, 3$. Once again, the summation w.r.t i and the phase factors w_i appear in eq. (4) for the 3D conformal field theory, which do not exist in that for the 2D conformal field theory in literature [1,3–5]. In eq. (4), however, because only the real part of the phase factors appears, the complex weight factors w_i are replaced by $|w_i|\mathrm{Re}|e^{i\phi_i}|$, where ϕ_i are phases [see Ref. [13] for details of topological phases in the 3D Ising model]. The famous Virasoro algebra still works, but here for each complex plane of quaternionic coordinates in the complexified quaternionic Hilbert space. The parameter c is the central charge, which is the same for different complex planes of a model, $c = 1/2$ for the Ising model. Note that if the theory contains a Virasoro field, the states transform in representation of the Virasoro algebra, rather than just of the Lie algebra of sl(2, C) corresponding to the Mobius transformation [4].

For the 2D conformal field theory, the scaling dimension is given by $\Delta = h + \bar{h}$, here h and $\bar{h}$ are known as the conformal weights of the state, which are independent (real) quantities. We have $h = \bar{h} = \frac{1}{16}$, and $\Delta = \frac{1}{8}$, and there is a relation between the scaling dimension Δ and the critical exponent η, based on the formalism of the correlation function: $\eta = 2\Delta + 2 - d$. For the 2D Ising model, $\eta = 1/4$ and $\Delta = 1/8$. For the 3D Ising model, the critical exponent η is found to be 1/8 in Ref. [9]. Thus, one has $\Delta = 9/16$.

Then, we discuss the physical essential of the phase factors w_i appearing in eq. (2) and eq. (4) above for the 3D conformal transformation. In the physics of gauge theories, Wilson lines correspond essentially to the space-time trajectory of a charged particle, i.e., so-called world histories of mesons or baryons [6]. Under a change of framing, the expectation values of Wilson lines are multiplied by a phase $e^{2i\pi h_a}$, where h_a is the conformal weight of the field. A twist of a Wilson line is equivalent to a phase, while a braiding of two Wilson lines from a trivalent vertex is also equivalent to a phase. The skein relation for Wilson lines in the defining N-dimensional representation of SU(N) can be found in Ref. [6]. In Ref. [9, 13, 14], Zhang proposed that the complex weight factors, i.e., the topological phase factors exist in the 3D Ising model. These topological phase factors may have the same origin of the phase in the expectation values of Wilson lines as those obtained by a change of framing in gauge theories. This is because the procedure for diagonalization involves the smoothing of knots/crossings in the 3D Ising system. Therefore, the phase factors w_i appearing in eq. (2) and eq. (4) above for the 3D conformal transformation may have the same origin as those in the expectation values of Wilson lines [as well as the 3D Ising model]. For further understanding, one may refer also to Ref. [23] for a nonadiabatic geometric phase in quaternionic Hilbert space, and Ref. [24] for a quaternionic phase and coherent states in quaternionic quantum mechanics.

6.

In 2D conformal field theory, canonical quantization on a circle S gives a physical Hilbert space H_s. A vector $\Psi \in \mathrm{H}_s$ is a suitable functional of appropriate fields on S, which corresponds to a local field operator O_Ψ. There is a relation in conformal field theory between vectors in the Hilbert space and local operators. A 3D analog of such relation between states and local operators can be found also, as shown in Ref. [6]. However, according to the quaternionic approach developed for the 3D Ising model, some new features are uncovered for relation between states and local operators in 3D systems. Usually, the physical Hilbert space obtained by quantization in $(2+1)$ dimensions can be interpreted as the space of conformal blocks in $(1+1)$ dimensions [25]. Analogously, the physical Hilbert space obtained by quantization in $(3+1)$ dimensions can be interpreted still as the space of conformal blocks in three $(1+1)$ dimensional complex planes of the quaternion coordinates.

We summarize here the procedure we developed above for treating the 3D conformal field theory:

1) Introducing an additional dimension to construct a $(3+1)$-dimensional framework to form the quaternionic coordinates.

2) Performing a unitary transformation, as a rotation in $(3+1)$-dimensions, to represent states and operators in the $(3+1)$-dimensional complexified quaternionic Hilbert space.

3) Introducing complex weight factors as topological phase factors w_i, for smoothing knots/crossings.

4) The decomposition of 3D conformal blocks to 2D conformal ones.

5) Dealing with 2D conformal blocks in each complex plane of quaternionic coordinates in the complexified quaternionic Hilbert space.

6) accounting the summation w.r.t. i of 2D conformal blocks in three complex planes together with the contributions of the phase factors w_i.

7.

Next, we discuss briefly the connection with the string theory: The world sheet is the 2D surface that the string sweeps out as it propagates through space-time, which can be described by a 2D-conformal field theory. Namely, conformal theories on the plane are often considered as string vacua, the nonfluctuating flat space version of some string theories, serving as toy models for quantum string theory [1]. The square Ising model on a closed Riemann surface with deformed squares, representing positive or negative curvature, is instrumental in investigations of quantum gravity of the coupling of a matter (Ising) theory to fluctuations of space-time geometry An interesting problem is how to understand the behaviour of the brane as it sweeps out in space-time.

Our understanding is: we need to describe of world volume of the brane in the $(3+1)$D framework as what one of us (ZDZ) did in Ref. [9] for the 3D Ising model. In this sense, suitable quaternionic coordinates with complexified weight factors should be introduced and then conformal invariance in three dimensions (actually in $(3+1)$D) can be utilized to study the behaviour of the world volume of the brane. This can be related with the observable universe, which could be a brane, i.e., a $(3+1)$ surface, embedded in the bulk, i.e, a $(3+1+d)$-dimensional spacetime, with standard-model particles and fields trapped on the brane, while gravity is free to access the bulk [26].

8.

In conclusion, quaternion-based functions developed in Ref. [9] for the 3D Ising model, which is related to the quaternionic sequence of Jordan algebras implied by the fundamental paper of Jordan, von Neumann, and Wigner [17], can be utilized to study the conformal invariance in dimensions higher than two. The 2D conformal field theory can be generalized to be appropriate for three dimensions, within the framework of the quaternionic coordinates with complex weights. The 3D conformal transformations can be decomposed into three 2D conformal transformations, where the Virasoro algebra still works, but only for each complex plane of quaternionic coordinates in the complexified quaternionic Hilbert space. Then one needs to perform the summation w.r.t. i of 2D conformal blocks in three complex planes together with the contributions of the phase factors w_i. Similar to 2D conformal field theories, local conformal invariance in 3D (though it is limited in each complex plane of quaternionic coordinates) enables exact solutions of 3D conformal field theories. The success of studying 3D critical systems, such as the 3D Ising model in Ref. [9], is due to this conformal invariance. The scaling dimension Δ is predicted to be $9/16$ for the 3D Ising model, to be contrasted with the known value $1/8$ for the 2D Ising model. We discuss the physical essential of the phase factors w_i appearing in eq. (2) and eq. (4) for the 3D conformal transformation, and suggest that they may have the same physical significance with those in the expectation values of Wilson lines [as well as the 3D Ising model]. Possibility of utilizing the present results for conformal invariance in three dimensions and quaternionic geometric phase in quaternionic Hilbert space for studying the behaviour of the world volume of the brane, or the world sheet of the string in 3D or $(3+1)$D, is briefly discussed. The present work is helpful for better understanding classical, conformal and topological field theories in high dimensions [27, 28].

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NIEZMIENNICZOŚĆ KONFOREMNA TRÓJWYMIAROWEGO (3D) MODELU ISINGA A KWATERNIONOWA FAZA GEOMETRYCZNA W KWATERNIONOWEJ PRZESTRZENI HILBERTA

S t r e s z c z e n i e

Bazując na kwaternionowym podejściu opracowanym przez jednego z nas [Z.-D. Zhang, Phil. Mag. **88** (2008), 3007] dla trójwymiarowego (3D) modelu Isinga, obecnie badamy konforemną niezmienniczość w trzech wymiarach. Wypracowujemy procedurę rozwijania 3D-konforemnej teorii pola. 2D-konforemna teoria pola jest rozszerzona tak, by stosowała się do trzech wymiarów w zakresie struktury współrzędnych kwaternionowych z wagami zespolonymi. Algebra Virasoro jest wciąż stosowalna, lecz dla każdej płaszczyzny zespolonej współrzędnych kwaternionowych. Kwaternionowe fazy geometryczne pojawiają się w kwaternionowej przestrzeni Hilberta w wyniku procedury diagonalizacji, w którą włączone jest wygładzanie węzłów/przecięć w wielowymiarowo oddziałującym 3D-układzie spinów Isinga. Jest również krótko rozważana możliwość stosowania konforemnej niezmienniczości w trzech wymiarach zachowania się objętości warstwy świata, względnie warstwy struny świata w modelu 3D lub $(3+1)$D.

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In memory of
Professor Roman Stanisław Ingarden

Jacques H.H. Perk

ERRONEOUS SOLUTION OF THREE-DIMENSIONAL (3D) SIMPLE ORTHORHOMBIC ISING LATTICES

Summary

Thirteen follow-up papers by Zhang and March perpetuate the errors of a 2007 paper by Zhang, which was based on an incorrect application of the Jordan-Wigner transformation and presents final expressions that contradict rigorously established exact results. The presentation given here can be used as a brief mathematical introduction to the Ising model for nonexperts.

Keywords and phrases: Ising lattice, quaternionic phase, quaternionic Hilbert space, conformal invariance in 3 dimensions

1. Introduction

In a very long paper [1] published in 2007 Z.-D. Zhang claims to present the exact solution of the free energy per site and of the spontaneous magnetization of the three-dimensional Ising model in the thermodynamic limit. This claim has been shown to be false [2–7] and we shall show here that very little original work, if any, in [1] can be salvaged.

The principal reason why the outcomes of [1] are wrong is that they contradict exactly known series expansion results [2, 5]. Several references were cited in [2, 5] which show that [1] violates rigorously established theorems. As these cited theorems are formulated for very general lattice models with rather general interactions, requiring complicated notations and such concepts as Banach spaces and Banach algebras, it takes some effort to check that every needed detail is there to make the proof rigorous.

Therefore, we present here a simpler self-contained presentation, restricted to the three-dimensional Ising model on a simple cubic lattice, which can be used as a short introduction for nonexperts interested in this model.

Definition 1. The *isotropic Ising model on* $\mathbb{Z}_n^3$, a periodic $n \times n \times n$ lattice with $N = n^3$ sites $i = (i_x, i_y, i_z)$ on a 3-torus, is defined by its configuration space

$$(1) \qquad \mathbb{T}^3 \supset \mathbb{Z}_n^3 \to \{\pm 1\}^N, \qquad i \mapsto \sigma_i = \pm 1, \text{ for } i \in \mathbb{Z}_n^3,$$

and its interaction energy

$$(2) \qquad \mathcal{H}_N : \mathbb{Z}_n^3 \to \mathbb{C}, \qquad \mathcal{H}_N = \mathcal{H}_N(\{\sigma_i\}) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i,$$

where the sum over $\langle i, j \rangle$ is over all nearest-neighbor pairs of sites i and j, J is the interaction strength and B is the scaled magnetic field. Sites i and j are nearest-neighbor (nn) sites, if and only if

$$(3) \qquad (i_x - j_x, i_y - j_y, i_z - j_z) = (\pm 1, 0, 0),\ (0, \pm 1, 0),\ \text{or}\ (0, 0, \pm 1) \bmod n.$$

Remark 1. The generalization to the orthorhombic lattice is straightforward, replacing n by n, n', n'' and J by J, J', J'' for the three lattice directions. We consider the isotropic lattice for the sake of simplicity of arguments, as this special case suffices to disprove Zhang's claims [1].

Definition 2. Given a function $A \equiv A(\{\sigma\})$ of the spin configuration, its *expectation value* is

$$(4) \qquad \langle A \rangle_N = \frac{1}{Z_N} \sum_{\{\sigma_i = \pm 1\}} A\, \mathrm{e}^{-\beta \mathcal{H}_N}, \qquad \langle A \rangle = \lim_{N \to \infty} \langle A \rangle_N,$$

where the *partition function*,

$$(5) \qquad Z_N = \sum_{\{\sigma_i = \pm 1\}} \mathrm{e}^{-\beta \mathcal{H}_N},$$

is a state sum taken over all 2^N spin configurations, while $\beta = (kT)^{-1}$ with T the absolute temperature and k Boltzmann's constant. If β, J, and B are real, then $\rho(\{\sigma\}) = \mathrm{e}^{-\beta \mathcal{H}_N} / Z_N$ is the Boltzmann-Gibbs canonical probability distribution.

Definition 3. The *free energy per site* f_N and its *infinite system limit* f are given by

$$(6) \qquad -\beta f_N = \frac{1}{N} \log Z_N, \qquad f = \lim_{N \to \infty} f_N,$$

whereas the spontaneous magnetization is defined by

$$(7) \qquad I = \lim_{B \downarrow 0} \lim_{N \to \infty} \langle \sigma_{i_0} \rangle_N = \lim_{B \downarrow 0} \lim_{N \to \infty} \frac{1}{Z_N} \sum_{\{\sigma_i = \pm 1\}} \sigma_{i_0}\, \mathrm{e}^{-\beta \mathcal{H}_N},$$

with i_0 any of the N lattice sites, as the lattice is chosen periodic. The pair-correlation function of spins at sites i and j is $\langle \sigma_i \sigma_j \rangle$.

Remark 2. As in [1], we shall concentrate on the zero-field ($B = 0$) thermodynamic limit ($\lim_{N\to\infty}$). The order of limits in (7) was used implicitly in Yang's paper [8] on the spontaneous magnetization of the square-lattice Ising model [9–11]. With the opposite order of limits the result is identically zero. An alternative definition is $I^2 = \lim\langle\sigma_i\sigma_j\rangle|_{B=0}$ in the limit of infinite separation of sites i and j [12].

In [1] Zhang starts out mimicking the treatment of the two-dimensional Ising model by Onsager and Kaufman [9–11], in order to calculate the free energy, magnetization and pair correlation of the three-dimensional case. Even though Zhang made two early errors in [1], while transforming to Clifford algebra operators and treating boundary terms [5], he claims [6] that these are overcome by two conjectures. But these conjectures are based on no serious evidence whatsoever and the resulting expressions for the free energy and magnetization [1] are demonstrably incorrect, as they fail the series test [2, 5].

First, in section 2, a detailed account will be given of the rigorous results of the 1960s violated by Zhang's work. Theorems 2 and 3 provide rigorous proof of the correctness of the series test. Then, in section 3, further comments will be presented, including several on the follow-up work by March and Zhang [13–25], which contain several additional errors and misleading statements.

2. Some rigorous results of the 1960s revisited

2.1. Free energy per site of a finite system vs. its large-system limit

In recognizing the criticisms to which his work in [1] has been subjected in [2, 4, 5, 7], Zhang (supported more recently by Norman H. March) has argued that the usual high-temperature series expansions [26], renormalization group treatments [27, 28], and Monte Carlo simulations [29, 30], fail to apply in the vicinity of infinite temperature owing to singular behavior and Yang–Lee zeros [31, 32] present even in the thermodynamic limit.

Hence, it is argued, such criticisms are not applicable as a basis for criticizing the quite different conclusions he has reached. See specifically the claims Zhang has made in the second paragraph of [3], and in the second half of page 766 of [6], as well in section 5 of [16], second half of page 534. The aim of this section is to show specifically by a detailed mathematical analysis that there is no credibility at all in these claims.

In fact, five decades ago several theorems were published and supported by fully rigorous proofs that underpin the validity of the criticisms of Zhang's work, see, e.g., [33, 34] for review. Nevertheless, let us here take the reader through a simplified treatment especially tailored to apply to the point at issue, namely, the statistical mechanics of the Ising model on a cubic lattice with periodic boundary conditions.

The proof of the thermodynamic (infinite system-size) limit of the free energy typically uses the following lemma, see e.g. (2.15) in [33]:

Lemma 1.

$$|\log \mathrm{Tr}\,\mathrm{e}^{A} - \log \mathrm{Tr}\,\mathrm{e}^{B}| \leq \|A - B\|, \quad \textit{for } A \textit{ and } B \textit{ hermitian.} \tag{8}$$

Proof. The proof follows immediately working out

$$\log \mathrm{Tr}\,\mathrm{e}^{A} - \log \mathrm{Tr}\,\mathrm{e}^{B} = \int_0^1 \frac{d}{dh} \log \mathrm{Tr}\,\mathrm{e}^{B+h(A-B)} dh = \int_0^1 \frac{\mathrm{Tr}\,(A-B)\mathrm{e}^{B+h(A-B)}}{\mathrm{Tr}\,\mathrm{e}^{B+h(A-B)}} dh,$$

where the last integrand is an expectation value. (In this paper we only need to consider the commuting case that A and B are diagonal matrices.)

Theorem 1. *The free energy per site f_N converges uniformly to a limit f as the system size becomes infinite for βJ real and bounded.*

Proof. In order to prove this we must estimate $|f_N - f_M|$ for $N, M > N_0$, with N_0 sufficiently large. Here we do that only for periodic cubic lattices $N = n^3$, $M = m^3$ and compare with the larger periodic cubic lattice of size $NM = (nm)^3$. By changing a subset of the interactions we can change the larger lattice into N identical cubes of the size M lattice, or the other way around. The proof is then provided by counting the changed interactions and by using Lemma 1. In our case, the trace in the lemma is just the sum over spin configurations and the norm the maximum over all configurations.

Remark 3. Lemma 1 can also be used to show that the free energy f does not depend on boundary conditions in the large system limit with different shapes than cubes, provided it is taken in the sense of van Hove, see e.g. [33, 34] for details.

Remark 4. The proof of Theorem 1 gives a rigorous bound on the difference of the free energy per site of a finite system and its large-system limit. It can therefore be used to estimate the accuracy of finite-size calculations using e.g. Monte Carlo simulations.

2.2. Analyticity of the correlation functions and their thermodynamic limits

Lemma 2. *The partition function Z_N* (5) *is a Laurent polynomial in $\mathrm{e}^{\beta J}$, so that βf_N is singular only for the zeros of this Laurent polynomial and for $\mathrm{e}^{\beta J} = \infty$. As Z_N is a sum of positive terms for real βJ, it cannot have zeros on the real axis.*

We will show that the zero closest to $\beta J = 0$ (or $\mathrm{e}^{\beta J} = 1$) in the complex βJ plane is uniformly bounded away, i.e. $Z_N \neq 0$ for all $|\beta J| < K_0$ and all N for some fixed K_0. This means that f_N can be expanded in a power series in βJ that is absolutely convergent for $|\beta J| < K_0$ and uniform in N. It is well known that more and more coefficients become independent of N as N increases. Together this implies that the limiting f also has a power series in βJ with radius of convergence at least K_0.

We continue by deriving a lower estimate for K_0. Most proofs of the analyticity of free energies and correlation functions use linear correlation identities of Schwinger–Dyson type, known under such names as the BBGKY hierarchy, Mayer–Montroll or Kirkwood–Salzburg equations. We could use [35] and [36]. But instead, let me give an alternative proof using an identity of Suzuki [37, 38], restricted to the isotropic Ising model on a simple cubic lattice with periodic boundary conditions and of arbitrary size, as this method also can be used to generate the coefficients of the high-temperature series. More precisely, using the canonical definition of the expectation value of a function $A \equiv A(\{\sigma\})$ of the spin configuration, we have the correlation identity [37, 38]:

Lemma 3. (M. Suzuki, 1965 [37, 38])

$$\left\langle \prod_{i=1}^{m} \sigma_{j_i} \right\rangle_N = \frac{1}{m} \sum_{k=1}^{m} \left\langle \left(\prod_{\substack{i=1 \\ i \neq k}}^{m} \sigma_{j_i} \right) \tanh\left(\beta J \sum_{l \text{ nn } j_k} \sigma_l \right) \right\rangle_N, \tag{9}$$

where $j_1, \ldots, j_m$ are the labels of m spins and l runs through the labels of the six spins that are nearest neighbors of σ_{j_k}.

Proof. The proof of (9) is easy summing over spin σ_{j_k} in the numerator of the expectation value, i.e.,

$$\sum_{\sigma_{j_k}=\pm 1} \sigma_{j_k} \mathrm{e}^{\beta J \sum_{l \text{ nn } j_k} \sigma_{j_k} \sigma_l} = \tanh\left(\beta J \sum_{l \text{ nn } j_k} \sigma_l \right) \sum_{\sigma_{j_k}=\pm 1} \mathrm{e}^{\beta J \sum_{l \text{ nn } j_k} \sigma_{j_k} \sigma_l}. \tag{10}$$

Averaging over k has been added in (9), so that all spins are treated equally, consistent with the periodic boundary conditions. The lemma is also valid without that.

Next we use

Lemma 4.

$$\tanh\left(\beta J \sum_{l=1}^{6} \sigma_l \right) = a_1 \sum_{(6)} \sigma_l + a_3 \sum_{(20)} \sigma_{l_1} \sigma_{l_2} \sigma_{l_3} + a_5 \sum_{(6)} \sigma_{l_1} \sigma_{l_2} \sigma_{l_3} \sigma_{l_4} \sigma_{l_5}, \tag{11}$$

where the sums are over the 6, 20, *or* 6 *choices of choosing* 1, 3, *or* 5 *spins from the given $\sigma_1, \ldots, \sigma_6$. It is easy to check that the coefficients a_i are*

$$a_1 = \frac{t(1 + 16t^2 + 46t^4 + 16t^6 + t^8)}{(1+t^2)(1+6t^2+t^4)(1+14t^2+t^4)}, \qquad a_3 = \frac{-2t^3}{(1+t^2)(1+14t^2+t^4)},$$

$$a_5 = \frac{16t^5}{(1+t^2)(1+6t^2+t^4)(1+14t^2+t^4)}, \qquad t \equiv \tanh(\beta J). \tag{12}$$

The poles of the a_i are at $t = \pm\mathrm{i}$, $t = \pm(\sqrt{2} \pm 1)\mathrm{i}$, and $t = \pm(\sqrt{3} \pm 2)\mathrm{i}$. It can also be verified, e.g. expanding the a_i in partial fractions, that the series expansions of the a_i in terms of the odd powers of t alternate in sign and converge absolutely as long as $|\beta J| < \arctan(2 - \sqrt{3}) = \pi/12$.

Proof. Clearly, the tanh in (11) can be expanded as done. Replacing all six spins, σ_l by $-\sigma_l$, shows that no terms with an even number of spins occur. Also, permutation symmetry allows only three different coefficients. Multiplying (11) with one, three, or five spins σ_l and then summing over all $2^6 = 64$ spin states, is one way to derive (12). It is then straightforward to verify the following partial fraction expansions,

$$\begin{aligned} a_{1,5} &= \frac{1}{24}\Big(\frac{p_1 t}{1+(p_1 t)^2} + \frac{p_2 t}{1+(p_2 t)^2}\Big) \pm \frac{\sqrt{2}}{8}\Big(\frac{p_3 t}{1+(p_3 t)^2} + \frac{p_4 t}{1+(p_4 t)^2}\Big) \\ &\quad + \frac{1}{3}\,\frac{p_5 t}{1+(p_5 t)^2}, \\ a_3 &= \frac{1}{24}\Big(\frac{p_1 t}{1+(p_1 t)^2} + \frac{p_2 t}{1+(p_2 t)^2}\Big) - \frac{1}{6}\,\frac{p_5 t}{1+(p_5 t)^2}, \end{aligned} \tag{13}$$

$$p_{1,2} = 2 \pm \sqrt{3}, \quad p_{3,4} = \sqrt{2} \pm 1, \quad p_5 = 1, \qquad (p_1 p_2 = p_3 p_4 = 1). \tag{14}$$

The remaining statements of the lemma follow from these expansions.

We can now prove the following two theorems for magnetic field $B = 0$:

Theorem 2. *The correlation functions $\langle\prod_{i=1}^{m}\sigma_{j_i}\rangle_N$ and their thermodynamic limits $\langle\prod_{i=1}^{m}\sigma_{j_i}\rangle$ are analytic, having series expansions in t or βJ with radius of convergence bounded below by (17) and uniformly convergent for all N including $N = \infty$. Let d be the largest edge of the minimal parallelepiped containing all sites $j_1, \ldots, j_m$. Then the coefficient of t^k with $k < n - d$ for the lattice with $N = n^3$ sites equals the corresponding coefficient for larger N, including the one for $N = \infty$.*

Proof. We can assume that $m > 0$ and even, since for m odd we have

$$\langle\prod_{i=1}^{m}\sigma_{j_i}\rangle_N \equiv 0$$

as it both is invariant and changes sign under the spin inversion $\sigma_i \to -\sigma_i$ for all sites i.

The system of equations (9)–(12) can be viewed as a linear operator on the vector space of linear combinations of all correlation functions of the 3-dimensional Ising model. It is easy to estimate the norm of this operator. Using the alternating sign property of the a_i's, it is easy to verify that a_1, a_3, and a_5 can all be written as t times a series in t^2, which three series consist of positive terms only when t is imaginary. This means that each $|a_i|$ is maximal for given $|t|$ when t is imaginary and within the radius of convergence, i.e. p_2 in (14).

From the $32m$ terms in the right-hand side (RHS) of (9) after applying (11), it follows then that we only need to study

$$6a_1 + 20a_3 + 6a_5 = \frac{2t(t^2+3)(3t^2+1)}{(1+t^2)(1+14t^2+t^4)} \tag{15}$$

for purely imaginary t to find the desired upper bound r for the norm. Setting $t = \mathrm{i}x$ with $0 < x < 2-\sqrt{3}$ to stay within the first pole of (15), we next define

$$r = \frac{2x(3-x^2)(1-3x^2)}{(1-x^2)(1-14x^2+x^4)}, \qquad \text{for } x = |t|. \tag{16}$$

We then have that the RHS of (9) is bounded by rM, where $M = \max|\langle\sigma\cdots\sigma\rangle|$ with the maximum taken over all $32n$ pair correlations in the RHS. (Obviously, $M \le 1$ if $\beta \ge 0$ and real, but we shall not use this.) We can easily show that $r < 1$ for

$$\begin{aligned} &|t| < (\sqrt{3}-\sqrt{2})(\sqrt{2}-1) = 0.131652497\cdots, \quad \text{or} \\ &|\beta J| < \arctan[(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)] = 0.130899693\cdots. \end{aligned} \tag{17}$$

To prove analyticity of $\langle\prod_{i=1}^{m}\sigma_{j_i}\rangle_N$ as a function of β at $\beta = 0$, we apply (9) to it. Then we apply (9) to each of the $32m$ new correlations, and we keep repeating this process ad infinitum. Since $\sigma_i^2 = 1$, we will from some point on regularly encounter the correlation with $m = 0$, i.e. zero σ factors, for which $\langle 1\rangle = 1$, so that the iteration process ends there. Each other correlation (with $m > 0$) vanishes with at least one power of t, as can be seen comparing e.g. (9) and (12). We conclude that the iteration process generates the high-temperature power series in t to higher and higher orders, for arbitrary given size N of the system.

To get the partial sum of the series to a given order, we only need to keep the contributions for which the iteration process has ended and expand all occurring a_i as series in t. The sum of the absolute values of the terms is bounded by $\sum r^j < \infty$ when (17) holds. However, the original correlation function is meromorphic with a finite number of poles away from the real t axis for any finite N. Thus for sufficiently high order of series expansion in t, the remainder term is arbitrarily small. The only possible conclusion is that we have proved convergence of the series expansion of $\langle\prod_{i=1}^{m}\sigma_{j_i}\rangle_N$ in powers of t, uniform in N with a finite radius of convergence in the complex t and β planes bounded below by (17).

To prove the final statement of the theorem for finite N, we notice that the above iteration process generates new correlations with the range of the positions j of the spins extended by one in a given direction. As long as we do not go around a cycle (periodic boundary condition) of the 3-torus, we do not notice any N-dependence. It takes at least $n - d$ iteration steps to notice the finite size of the lattice.

Combining the convergence uniform in N with the fact that more and more coefficients converge with increasing N, we conclude that $\langle\prod_{i=1}^{m}\sigma_{j_i}\rangle_N$ converges to a unique limit as $N \to \infty$ for $|t| < 2-\sqrt{3}$, with the properties stated in the theorem.

2.3. The reduced free energy and its thermodynamic limit

Theorem 3. *The reduced free energy βf_N for arbitrary N and its thermodynamic limit βf are analytic in βJ for sufficiently high temperatures. They have series expansions in t or βJ with radius of convergence bounded below by* (17) *and uniformly convergent for all N including $N = \infty$. The first $n-1$ coefficients of these series for $N = n^3$ equal their limiting values for $N = \infty$.*

Proof. To prove analyticity of βf in terms of β at $\beta = 0$ it suffices to study the internal energy per site or the nearest-neighbor pair correlation function, as

$$u_N = \frac{1}{N}\langle \mathcal{H}_N \rangle_N = \frac{\partial(\beta f_N)}{\partial \beta} = -3J\langle \sigma_{0,0,0}\sigma_{1,0,0}\rangle_N, \tag{18}$$

as follows from (5) and (6). Here $\sigma_{0,0,0}$ and $\sigma_{1,0,0}$ can be any other pair of neighboring spins. The proof then follows from Theorem 2 and integrating the series for u_N, using $Z_N|_{\beta=0} = 2^N$, implying $\lim_{\beta\to 0} \beta f_N = -\log 2$.

Remark 4. Adding a small magnetic field B and generalizing the steps in the above, we can conclude that all correlation functions are finite for small enough $|\beta|$ and $|\beta H|$, so that there are no Yang–Lee zeros [31, 32] near the $H = 0$ axis for small β and H. The proof can also be generalized to the case that the interactions are anisotropic, i.e. J, J', J'' as in [1]. Then Z_N is a Laurent polynomial in each of $\mathrm{e}^{\beta J}$, $\mathrm{e}^{\beta J'}$, and $\mathrm{e}^{\beta J''}$, etc.

Remark 5. Similar results can be derived for the low-temperature series, for example after applying the Kramers–Wannier duality transform to the high-temperature regime of the dual system with spins in the centers of the original cubes and with four-spin interactions around all cube faces perpendicular to the edges of the original lattice [39].

Remark 6. It is straightforward to calculate the first few high-temperature series coefficients of the free energy by the method described in this section, with or without using the averaging in (9). They agree with the long series reported in [26] and earlier works cited there. Zhang's free energy formula claimed for all finite temperatures [1] does not agree, as already the coefficients of $\kappa^2 \equiv t^2$ in (A12) and (A13) of [1] differ. Zhang's excuse that there are two expansions, one for finite β and one for infinitesimal β, violates general theorems, that apply to more general models than the Ising model [4, 5]. Here this excuse is invalidated in detail by Theorem 3.

Remark 7. Zhang's spontaneous magnetization series is obviously wrong. In three dimensions one should have $I - 1 = \mathrm{O}(x^6)$, with $x \equiv \mathrm{e}^{-\beta J}$ in the low-temperature limit, $x \to 0$ ($J > 0$), as each spin has six nearest neighbors [5], rather than eight, which would result in the four-dimensional $I - 1 = \mathrm{O}(x^8)$ presented by Zhang in (103) of [1].

Remark 8. The finite radius of convergence of the series expansions about $\beta = 0$ is also hinted at by the fact that the zeros of Z_N for $T = \infty$ occur for $B = \pm \mathrm{i}\infty$, $\beta B = \pm \mathrm{i}\pi/2$ [7]. For fixed temperature T or $\beta = 1/kT$ and $\beta J > 0$ real the zeros of Z_N lie on the unit circle in the complex $\mathrm{e}^{-2\beta B}$ plane [32], all located at -1 at infinite temperature [7] and spreading out with decreasing temperature until the zeros "pinch" $+1$ on both sides of the unit circle at and below the critical temperature, in agreement with the theory of Yang and Lee [31, 32]. Zhang's claim that this pinching at $+1$ also occurs at $\beta = 0$ [1, 3, 6, 16] is disproved by Theorem 3.

Corollary 1. *As pointed out already in* [2, 4, 5]*, all final results of* [1] *are proven wrong, as they do not agree within a finite radius of convergence with the well-known series expansion coefficients. This also means that the conjectures of* [1] *are falsified.*

3. Further remarks and objections

3.1. Two series expansions for the same object

In appendix A of [1] Zhang claims to reproduce the first 22 terms of the high-temperature series for the free energy. But this is no more than reverse engineering, fitting the known coefficients [26] to an integral transform (A.1) or (74) in [1] giving the first few coefficients of the weight functions as given in (A.2). There is no more information than the series results provided by others, so that this does not constitute a new result, as explained in [2, 5].

As this construction this way is based on a conjectured integral transform of weight functions that can only be reconstructed from a few known series coefficients, it cannot be considered an exact solution. Knowing this, Zhang conjectures ad hoc above (A.3) on page 5400 another choice for the weight functions, namely $w_x = 1$, $w_y = w_z = 0$, leading to another high-temperature series for non-infinitesimal temperatures, in violation of the rigorous result on the uniqueness of the series expansion presented in section 2. This is not sound mathematics [2].

3.2. Citations by other authors

The outcomes of [1] have been criticized in [29, 30], as they disagree with recent high-precision Monte Carlo calculations presented there. Both the position of the critical point and the values of the critical exponents differ from the ones in [1], while the results of [29, 30] agree with those of many others obtained by a variety of methods [28].

One paper on a decorated three-dimensional Ising model [40], mapping this model exactly to the Ising model on a cubic lattice, used Zhang's free energy [1] as an approximate result in the analysis. The experimental paper [41] states that their result for the critical exponent $\Delta = 2.0 \pm 0.5$ is consistent with [1]. However, the reported error bar is so large that this means nothing. Moreover, paper [42] on the Heisenberg model only briefly cites [1] as an Ising reference.

The authors of [43, 44] learned from [1] the quaternion setup of the transfer matrix of the three-dimensional Ising model, which was well-known earlier, see e.g. [45–47].[1] In Zhang's work [1] this is treated before the first error occurs with the Jordan–Wigner transformation to Clifford algebra operators. His P's and Q's do not anticommute [5, 6].

3.3. Advertising wrong critical exponents

Klein and March took the exact Ising critical exponents for dimensions $d = 1, 2, 4$ together with the proposal of [1] for $d = 3$ and made an ad hoc fit [48] for all real $1 \leq d \leq 4$. However, they failed to compare with the results from ε-expansion [27,28], where $\varepsilon = 4 - d$. This is a serious shortcoming, as the [48] formulae disagree with the ε-expansion exponents for small ε and fail the one foremost explicit test available. It may also be noted that the extrapolated Ising exponents for $d = 3$ from ε-expansion agree with those extracted from series expansions and Monte Carlo calculations [28], while differing from those presented in [1].

March and Zhang have followed this paper [48] up with thirteen publications, thus perpetuating the errors of the original work [1]. Some of these works compare Zhang's critical exponents with those from experiments on $CrBr_3$ and Ni [13, 14]. Nickel is known to have Heisenberg exchange interactions and its critical exponent β is about the accepted value for the three-dimensional Heisenberg model, which is also about Zhang's value wrongly claimed for Ising.

Comparing experiments with models needs a discussion of the interactions in the experimental compounds, whether Ising or Heisenberg, isotropic or anisotropic, short-range or long-range, etc. No such analysis was presented. The same objection can be brought up about section 2 of [20].

In [15] critical exponents for the two- and three-dimensional q-state Potts model are discussed. Those for $d = 2$ are by now well established, but the values presented for $d = 3$ cannot all be correct, as for the Ising case $q = 2$ the exponents of [1] have been used.

In [18,19] a new formula for critical exponent δ is given, improving the one in [48]. The same objection still applies, as again no comparison with ε-expansion is made.

It is implied by the theory of Yang and Lee [31, 32], that the best experimental results on Ising exponents are to be expected from measurements on liquid-gas transitions in simple substances. March and Zhang have admitted that the exponents of [1] fail this test, see section 3 of [20]. Their suggestion that the experiment needs to be redone carries no credibility, as the critical exponents measured in a number of similar experiments are indeed typical for Ising, see section 3.2.2 of [28].

[1] Maddox only published his final formula for the free energy [45]. The details were discussed in a special session, where also his error (the same as Zhang's first error [5]) was discovered.

3.4. Singularity of free energy at $T = \infty$

Several statements in section 5 of [16] repeat and expand on statements in [1, 3, 6] contradicting rigorous theorems discussed in section 2 above. For example, while it is correct that the free energy f diverges at $T = \infty$, this does not correspond to a physical singularity, as the combination βf is to be used. Indeed, $\mathrm{e}^{-\beta f}$ relates to the normalization of the Gibbs ensemble probability distribution and βf is the principal object of Theorem 3. Multiplying βf with kT results in f having a convergent Laurent expansion with a leading pole at $T = \infty$ that has no physical significance. Another point is discussed in Remark 2.3 in section 1.

3.5. False argument for $\alpha = 0$

Paper [17] addresses tricritical behavior. The authors claim that the logarithmic divergence of the specific heat, $\alpha = 0$ (log) at tricritical points in three dimensions, supports the similar value reported in [1] for the Ising critical behavior. However, this reasoning is flawed lacking any theoretical basis and contradicts the accepted value $\alpha = 0.110 \pm 0.001$, see eq. (3.2) and tables 3–7 of [28].

3.6. The $\epsilon = d - 2$ expansion

In papers [21, 22] on Anderson localization the authors say that $\epsilon = d - 2$ is not a small parameter for $d = 3$, just like $\varepsilon = 4 - d$ of the ε-expansion is not. This ignores that the best ε-expansion extrapolation results agree remarkably well with those from series, Monte Carlo, and experiment [28]. This argument to support [1] is again not valid.

3.7. Higher dimensions

The combinatorial sums defining the 3-dimensional Ising model involve commuting spin variables and an interaction energy that is a function of these spins. There is no reason to introduce time and quantum mechanics in this classical system, as is done in [23]. On the other hand, introducing the transfer matrix changes one space coordinate to (discrete) imaginary time. After "Wick rotation" to real time the 3-dimensional Ising model relates to a (2+1)-dimensional quantum system. The fourth dimension introduced in [1] has only been used to obtain wrong results violating rigorous results.

3.8. Fractal dimensions based on wrong results

In [24] Zhang and March write down some proposals for fractal dimensions. However, the values given for dimension 3 are based on incorrect results of [1].

3.9. Unfounded Virasoro algebra

The most recent paper [25] uses the weight factors of [1] to introduce a Virasoro algebra in 3+1 dimensions. This is ad hoc and the notations in equation (5) and seven lines below (6) there are not mathematically sound. To take the real part of the absolute value of a phase factor instead of just writing 1 makes no sense. Also, the Virasoro algebra relates to an infinite dimensional symmetry, which is only consistent with conformal symmetry in two dimensions, see e.g. [49]. Therefore, [25] has fundamental errors.

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BŁĘDNE ROZWIĄZANIE TRÓJWYMIAROWYCH (3D) PROSTOPADŁOŚCIENNYCH SIATEK ISINGA

S t r e s z c z e n i e

Trzynaście następujących po sobie prac Zhanga i Marcha powtarzało błędy popełnione w pracy Zhanga z roku 2007 oparte na nieprawidłowym zastosowaniu transformacjii Jordana-Wignera i prezentowało końcowe wyniki sprzeczne z precyzyjnie uzyskanymi rezultatami. Obecna prezentacja może być traktowana jako krótkie wprowadzenie matematyczne do modelu Isinga dla nie-ekspertów.

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Zhidong Zhang and Norman H. March

CONFORMAL INVARIANCE IN THE THREE-DIMENSIONAL (3D) ISING MODEL AND QUATERNIONIC GEOMETRIC PHASE IN QUATERNIONIC HILBERT SPACE II

RESPONSE TO "ERRONEOUS SOLUTION OF THREE-DIMENSIONAL (3D) SIMPLE ORTHORHOMBIC ISING LATTICES" BY J. H. H. PERK

Summary

This paper is a Response to Professor Perk's recent Comment (arXiv:1209.0731 and the preceding article in this journal). We point out that the singularities of the reduced free energy βf, the free energy per site f and the free energy F of the 3D Ising model differ at $\beta = 0$. The rigorous proof presented in the Perk's Comment is only for the analyticity of the reduced free energy βf, which loses its definition at $\beta = 0$. Therefore, all of his objections lose the mathematical basis, which are thoroughly disproved. This means that the series expansions cannot serve as a standard for judging the correctness of the exact solution of the 3D Ising model. Furthermore, we note that there have been no comments on the topology-based approach developed by Zhang for the exact solution of the 3D Ising model.

Keywords and phrases: Ising lattice, quaternionic phase, quaternionic Hilbert space, conformal invariance in 3 dimensions

1.

This paper is a Response to Professor Perk's recent Comment [1] on the exact solution of the three-dimensional (3D) Ising model, derived by one of us (ZDZ) based on two conjectures [2], and our recent paper on conformal invariance in the 3D Ising model [3] (and other earlier published papers). Firstly, we would like to point out that it is hard for us to find anything new in this new Comment, comparing with his three-years-old Comment/Rejoinder (and arXiv posting) [4, 5]. He repeats his objections published in [4, 5], though this time they have been formulated in more mathematical forms. We think that in this paper, it is unnecessary to repeat all of Zhang's responses in [6,7], and to refute point-to-point Perk's objections to follow-up papers by Zhang and March.

One of the main objections Perk repeated is that the solution obtained in [2] was wrong, because it was based on an incorrect application of the Jordan-Wigner transformation. However, as Zhang responded in [7], this error in the application of the Jordan-Wigner transformation does not affect the validity of the putative exact solution, since the solution is not derived directly from it. The Conjecture 1, i.e., the approach proposed in [2] for dealing with the topological problems of the 3D Ising model by a rotation in an additional dimension, can be applied directly to the corrected formulation after the correct application of the Jordan-Wigner transformation. We emphasize here that the Conjecture 1 serves for the topologic problems existing in the 3D Ising model (not for the incorrect formula published in [2]). There have been no comments on this topology-based approach underlying the derivation yet.

2.

Other main objections in Perk's recent Comment [1] and his previously published Comment/Rejoinder [4,5] are limited to the outcome of the calculations in [2]. These objections are based on a misjudgment that the exact solution of the 3D Ising model must pass the series test, and the solution found in [2] contradicts so-called exactly known series expansion results. Such argument is based on a belief that there are rigorously established theorems for the convergence of the high-temperature series. However, as pointed out already in [6, 7], all the well-known theorems for the convergence of the high-temperature series are rigorously proved only for β $(= 1/k_BT) > 0$, not for infinite temperature ($\beta = 0$). Exactly infinite temperature has been never touched in those theorems cited in [1, 4, 5, 8] for the free energy per site f of general lattice models with general interactions, since there is a possibility of the existence of a singularity at $\beta = 0$. For instance, Lebowitz and Penrose indicated clearly in p. 102 of their paper [9] that there is no general reason to expect a series expansion of p or n in powers of β to converge, since $\beta = 0$ lies at the boundary of the region E of (β, z) space. Their proof includes $\beta = 0$ only for hard-core potential in section II of their paper [9], not for the Ising model discussed in other sections. Lebowitz and Penrose at the end of the section II used a word of 'implies' as referred to Gallavotti et al.'s work [10]. However, although Gallavotti et al. proved that the radius of convergence is greater than zero, but once again their proof does not touch $\beta = 0$ [10], since the inequality just above (1), i.e.,

$$\sum_{T \cap X - \phi,\ T \neq \phi} |K_{\beta\phi'}(X, T)| \leq [\exp(e^{\beta\|\phi'\|} - 1) - 1]$$

is invalid for $\beta = 0$.

Although it is evident that the general theorems above fail in a rigorous proof for the convergence of the high-temperature series at $\beta = 0$, Perk still insists to carry out a proof restricted to the 3D Ising model on a simple cubic lattice [1]. In [1], he made an effort to prove rigorously Theorem 2.9, which is then used as a sword to disprove the exact solution of [2]. Unfortunately, this sword is made of wax! Perk claimed that Theorem 2.9 proves rigorously the analyticity of the reduced free energy βf in terms of β at $\beta = 0$. However, as will be described in details below, the reduced free energy βf loses its definition at $\beta = 0$, and furthermore, it has different behaviour with the free energy per site f. Actually, some mathematical tricks have been performed carefully in his procedure, in order to cover the truth that the analyticity of the free energy per site f in terms of β at $\beta = 0$ cannot be proved rigorously. Such tricks first appear in Definition 1.4, which defines the free energy per site f_N and its infinite system limit f by Eq. (6), but in form of $-\beta f_N$. Then Lamma 2.5 goes on perpetrating the fraud to discuss the singularity of βf_N, and finally to prove Theorem 2.9 'rigorously' for βf.

3.

In what follows, we will discuss in detail the singularities of the free energy at/near infinite temperature (see also [7] and its arXiv posting 0812.0194). The key issue here is that the behaviours of the reduced free energy βf, the free energy per site f and the free energy F differ at $\beta = 0$. Furthermore, both the reduced free energy βf and the free energy per site f lose their definitions at $\beta = 0$ so that one has to face directly the behaviour of the free energy F at $\beta = 0$.

Let us start from the initial point of the problem to discuss the origin of the singularities at/near infinite temperature. The total free energy of the system is:

$$F = U - TS = -k_B T \ln Z.$$

The singularities in the free energy and other thermodynamic consequences (such as the entropy, the internal energy, the specific heat, the spontaneous magnetization, etc.) originate from the singularities of the partition function Z. This is why Yang and Lee discussed the phase transition by evaluating the distribution of roots of the grand partition function (i.e., $Z = 0$) in their general theory [11, 12]. In order to describe infinite systems, one usually normalizes the extensive variables that are homogeneous of degree one in the volume, by the volume V (or the number of particles N), keeps the density (i.e. the number of particles per volume) fixed and takes the limit for V (or N) tending to infinity. In this sense, one usually defines the thermodynamic limit ($N \to \infty$) for the free energy per site f by

$$f = F/N = -k_B T \ln \lambda \quad \text{with} \quad \lambda = Z^{1/N}.$$

By such a procedure, it is expected that one can establish the fact that f converges uniformly to its common limit as $N \to \infty$, namely, it is performed with an assumption (or expectation) that f is finite [11–14]. In this way, one can easily avoid to deal with the total free energy $F = -Nk_BT \ln \lambda$ of the system, which shows singularities at any temperature as $N \to \infty$ and if $\ln \lambda$ is finite. However, at infinite temperature ($T = \infty$), there still exists a singularity in the free energy per site f that is equal to negative infinite in the case that $\ln \lambda$ is positive and finite. Using the value of the 3D Ising model $\lambda = 2$, one also finds that

$$f = -k_BT \ln 2 = -k_B \ln 2^T$$

has a singularity at $T = \infty$. This is inconsistent with the assumption for the definition of the free energy per site f, and therefore, such definition loses physical significance at $T = \infty$. This fact also indicates clearly that $\beta = 0$ is a special point, differing with other temperatures. It is clear that one has to face directly the total free energy F to study the singularities of the system at $T = \infty$.

4.

The total free energy of the system can also be written as $F = k_BT \ln Z^{-1}$. Therefore, besides the roots of the partition function Z, one should also discuss the roots of Z^{-1}. Writing $z \equiv \exp(-2\beta H)$ and keeping βH fixed in the limit $\beta \to 0$, the partition function of an arbitrary lattice with N sites for the Ising model becomes $Z = (z^{1/2} + z^{-1/2})^N$ [5]. It is easily seen that $z^{1/2} + z^{-1/2} > 1$ satisfies the condition for the zeros of the reciprocal of the partition function, i.e.,

$$Z^{-1} = (z^{1/2} + z^{-1/2})^{-N}.$$

So, the infinite-temperature zeros of Z^{-1}, i.e., $Z^{-1} \to 0$, occur at $z = 1$ as $N \to \infty$, $Z = 2^N \to \infty$. Or more explicitly speaking, the zeros are located at $\beta = 0, z = 1$. This point of view can be supported by the fact that the singularity behavior of the logarithmic function $\ln x$ in the two cases of $x = 0$ and $x = \infty$ correspond to those in the logarithmic function $\ln y$ with $y = 1/x$ in two cases of $y = \infty$ and $y = 0$, respectively. It indicates that both singularities at the two limits of $Z = 0$ and $Z = \infty$ are actually the same, except for a minus sign, and considerable interest should be paid to both of them. In Definition 1.4 of [1], the negative sign was carefully moved to the left-hand-side of Eq. (6), to avoid the discussion on zeros of Z^{-1}. But, if one would always try to conceal singularities of $\ln Z^{-1}$ by mathematical tricks, one would find similar tricks to remove singularities of $\ln Z$ also to violate the Yang-Lee Theorem [11, 12]. The singularities of the free energy F and the free energy per site f at $\beta = 0$ suggests that two different forms could exist for the high-temperature series expansions of the free energy per site f.

Perk argued in [5] that such singularities of the whole system are not of physical significance, which should be removed by using the reduced free energy per site βf. Perk now admits in page 10 of [1] that the free energy per site f diverges at $T = \infty$, but he still insists that this does not correspond to a physical singularity, as the combination βf is to be used. So, it is important to evaluate whether one can use the reduced free energy βf at $T = \infty$. As stated in Perk's Rejoinder (and arXiv posting) [5], the reduced free energy per site βf is often rewritten as $\beta f = \phi(\{K_i\}, h) = \phi(\{\beta J_i\}, \ \beta H)$ with some function ϕ. But the error in [1, 5] is easily seen as follows: One needs to set $\beta = 1$ to reach an equivalent between βf and f. Setting $\beta = 1$ equalizes to $T = 1/k_B \neq \infty$. Therefore, the necessary and sufficient condition for using the dimensionless parameters $K_i = \beta J_i$, $(i = 1, 2, 3)$ and $h = \beta H$ and setting $\beta = 1$ is $\beta \neq 0$. Thus, setting $\beta = 1$ is loss of generality for $\beta = 0$, and the replacements $J_i \to \beta J_i, \ H \to \beta H$ and $f \to \beta f$ are validated only for $\beta \to 0$ (not for $\beta = 0$). Clearly, all discussions in the Perk's Rejoinder [5] and recent Comment [1] for the reduced free energy βf are only valid at the limit $\beta \to 0$, but not 'exactly' at infinite temperature ($\beta = 0$). Clearly, the intrinsic characters of singularities of the zero at infinite temperature are quite different from those at finite temperatures, which cannot be disregarded by the usual process of removing the singularity at finite temperatures by using $-\beta f$.

5.

From the Yang-Lee Theorem [11, 12] and the findings above, in the 3D Ising model there indeed exist three singularities:

1) $H = 0$, $\beta = \beta_c$;
2) $H = \pm i\infty$, $\beta \to 0$;
3) $H = 0$, $\beta = 0$.

The third singularity is usually concealed in literature by setting $Z^{1/N}$ and dividing the total free energy F by N (equally, disregarding the singularity of zeros of Z^{-1}). The point of $\beta = 0$ has been avoided during the procedure of rigorous proof of all the previous theorems for the analyticity of the free energy per site f and also for the convergence of the high-temperature series. The difficulty has been bypassed by using the dimensionless parameters $K_i = \beta J_i$, $(i = 1, 2, 3)$ and $h = \beta H$ and setting $\beta = 1$. We point out here that the third singularity has physical significance: The 3D Ising system experiences a change from a 'non-interaction' state at $\beta = 0$ to an interacting state at $\beta > 0$. This change of the states is similar to a 'switch' turning off/on all the interactions at/near infinite temperature, resulting in the change of the topologic structures and the corresponding phase factors [2, 6, 7]. The topologic difference of $\beta = 0$ and $\beta \to 0$ requires the different dimensions (3D and $(3+1)$D, respectively) for describing the many-body interacting Ising system. These also support that the high-temperature series expansions of the free energy per site f can have two different forms for infinite temperature and finite temperatures, as revealed in [2].

In Section 3 of [1], Perk raised some further remarks and objections on other follow-up papers of [2]. We rebut these criticisms briefly as follows:

The transfer matrix V for the 3D Ising model consists of two kinds of contributions: those reflecting the local arrangement of spins and others reflecting the non-local behaviour of the knots. Any procedure (like, low- and high-temperature expansions, Monte Carlo method, expansions, renormalization group, etc.), which takes only the local spin configurations into account (without topological contributions), cannot be correct for the 3D Ising model. This is because the global (topological) effect exists in the 3D Ising system so that the flopping of a spin will sensitively affect the alignment of another spin located far from it (even with infinite distance). These approximation methods have close relations with the same shortcomings, and thus they obtain the close results, but all far from the exact solution. If one used the exact solution of [2] as a standard, the difference between the exact solution and these approximation approaches would be the good evaluation of the non-local contributions of the 3D Ising model. Whether the exact solution can predict the unknown terms of the usual high-temperature expansions is not important, since such expansions are valid only at temperatures very close to infinite temperature, where the global effect can be neglected. The important thing here is that the exact solution can predict all the terms of another high-temperature series for all finite temperatures, which take into account the topologic contribution of knots (internal factors) in transfer matrixes. On the other hand, the exact solution of the 3D Ising model does not need to fit with the low-temperature series. This is because the low-temperature series diverges, which suggests that it is falsified and the validity of its leading term is also doubted. The lack of information of the global behaviours of the 3D Ising system is the root of such divergence in the falsified well-known low-temperature series.

6.

It is true that after "Wick rotation" to real time the 3D Ising model relates to a (2+1)-dimensional quantum system. But what we proposed in [2, 3, 6, 7, 15] is more than "Wick rotation", and we introduced the fourth dimension to deal with the topological problems of the 3D Ising model, which agrees well with the topological theory. The introduction of the fourth dimension is also important for the time average [6, 7, 15] and for quaternionic Hilbert space with quaternionic geometric phase [2, 3]. The quaternionic form developed in [2] for wave functions of the 3D Ising model agrees well with Jordan-von Neumann-Wigner procedure [16] according to [17–20], and relates closely with well-developed theories, for instance, complexified quaternion [21], quaternionic quantum mechanics [22–24], and quaternion and special relativity [25]. The quaternion-based functions developed in [2] for the 3D Ising models can be utilized to study the conformal invariance in dimensions higher than two [3]. The 2D conformal field theory can be generalized to be appropriate for three dimensions, within the framework of the quaternionic coordinates with complex weights. The 3D conformal transformations can be decomposed into three 2D conformal transformations, where the Virasoro algebra still works in 2D, but only for each 2D complex plane of quaternionic coordinates in the complexified quaternionic Hilbert space [3]. Finally, we note that it is difficult to obtain the high accuracy of numerical results by Monte Carlo method and renormalization group, due to the limited capability of computers dealing with the 3D Ising or Heisenberg model with global effects in thermodynamic limit (infinite) systems (with 2^N configurations for Ising model as N $\to \infty$; even much more configurations for Heisenberg model). Thus it should be very careful to judge which system is of Ising or Heisenberg-type based on experimental and numerical results. Furthermore, it is well-known that one cannot distinguish the curves with power exponent $\alpha < 0.2$ and logarithmic exponent $\alpha = 0$, within errorbars of experiments and numerical calculations [2].

7.

In summary, we have disproved the Perk's recent Comment [1]. We have shown that both f and βf lose their definitions at $\beta = 0$, but with different consequences: the free energy per site f could have two different forms for the high-temperature series expansions as revealed in [2]; the reduced free energy per site βf can be used only for finite temperatures ($\beta > 0$), not for exactly infinite temperature ($\beta = 0$). The Perk's objections [1] are based on errors of mixing the concepts $T \to \infty$ and $T = \infty$ (i.e., $\beta \to 0$ and $\beta = 0$), and βf and f. His rigorous proof in [1] is only for the analyticity of the reduced free energy βf (its validity is held at $\beta \to 0$), not of the free energy per site f. Therefore, all the objections of Perk's Comment [1] do not stand on solid ground and have been rejected. The series expansions cannot serve as a standard for disproving the exact solution found in [2] of the 3D Ising model. Furthermore, there have been no comments on the topology-based approach developed by Zhang in [2] for the exact solution of the 3D Ising model.

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NIEZMIENNICZOŚĆ KONFOREMNA TRÓJWYMIAROWEGO (3D) MODELU ISINGA A KWATERNIOWA FAZA GEOMETRYCZNA W KWATERNIONOWEJ PRZESTRZENI HILBERTA II

ODPOWIEDŹ NA "BŁĘDNE ROZWIĄZANIE TRÓJWYMIAROWYCH (3D) PROSTOPADŁOŚCIENNYCH SIATEK ISINGA" J. H. H. PERKA

S t r e s z c z e n i e

Praca stanowi odpowiedź na niedawny Komentarz Profesora Perka (arXiv: 1209.0731 i poprzedni artykuł w niniejszym czasopiśmie). Zwracamy uwagę na fakt, że osobliwości zredukowanej energii swobodnej βf, energii swobodnej odniesionej do miejsca f oraz energii swobodnej F w modelu Isinga 3D różnią się dla $\beta = 0$. Rygorystyczny dowód zamieszczony w Komentarzu Perka stosuje się jedynie do analityczności zredukowanej energii swobodnej βf, która staje się niezdefiniowana dla $\beta = 0$. Tak więc wszystkie jego obiekcje tracą matematyczną podstawę, która daje się całkowicie obalić. Oznacza to, że rozwinięcia w szeregi nie mogą być miarodajne do osądzania poprawności dokładnego rozwiązania w modelu Isinga 3D. Co więcej, zauważamy, że nie zostało skomentowane oparte na topologii podejście Zhanga dotyczące dokładnego rozwiązania w modelu Isinga 3D.

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Jacques H.H. Perk

ERRONEOUS SOLUTION OF THREE-DIMENSIONAL (3D) SIMPLE ORTHORHOMBIC ISING LATTICES II

COMMENT TO THE RESPONSE TO "ERRONEOUS SOLUTION OF THREE--DIMENSIONAL (3D) SIMPLE ORTHORHOMBIC ISING LATTICES" BY Z.-D. ZHANG

Summary

The response by Zhang and March to a recent comment on several of their papers only adds further errors and misleading statements.

Keywords and phrases: Ising lattice, quaternionic phase, quaternionic Hilbert space, conformal invariance in 3 dimensions

1. The series test is decisive

In their Response [1] to my Comment [2] Zhang and March do not really address the new criticism raised to their work. Contrary to what is said, Comment [2] contains very little that is in my earlier Comment [3] and Rejoinder [4]. There is some material from the unpublished additions to the arXiv version of the Rejoinder, but that part is much improved with several new details added in [2]. There are also some pages discussing papers published later. Therefore, the statement "hard to find anything new" is wrong. Also, several statements are not even addressed in the Response and cannot be covered with the "unnecessary to repeat all of Zhang's responses".

Zhang seems to demand that I only comment on the "validity of the topologic approach developed", even though this is not precisely defined in any of his papers, apart from the formulation of his two conjectures in [5]. However, these conjectures 1 and 2 are not backed up by any quantitative evidence in the original 117 page work. Their validity can at this moment only be judged by the resulting free energy.

In section 3.1 of [2], I noted that Zhang expresses the free energy by an integral transform, given in (49) in [5], on unknown weight functions w_x, w_y, w_z, without a clear convincing argument how to get these weight functions. This is brought as a consequence of conjectures 1 and 2 [5] and it is analogous to saying that the free energy is a Fourier transform of some unknown function, by itself an empty statement.

Zhang made two choices in [5]. The first one is fitting series (A2) in [5] to the free-energy high-temperature series to as many terms as are known in the literature; the other is choosing weights (1,0,0). The first way gives no exact result, as one has no more than the known series terms. The second way leads to a different series, with the first nontrivial term differing from the known series; it is disproved by the first few terms of the well-known high-temperature series, since these have been rigorously established, also by the construction in [2].

The older proofs cited in [3, 6] are correct but not easy to read. Therefore, I gave a much simpler proof with mathematical precision in [2]. My proof does not depend on the papers by Lebowitz and Penrose [7] and by Gallavotti et al. [8], contrary to what Zhang seems to suggest.

2. Arguments for phase transition at $T = \infty$ are invalid

Statements made in [1] about [7,8] are taken out of context. The inequality Re$\beta > 0$ on page 102 in [7] is needed when the gas model has no hard core. Section II, however, opens with the statement that analyticity at $\beta = 0$ can be shown for a hard core potential. The Ising model is equivalent to a lattice gas version with at most one particle per lattice site (empty-occupied becomes spin $+/-1$), a special case of a hard core on the lattice. Thus the objection that [7] excludes $\beta = 0$ in their analyticity proof does not apply.

One statement in [1] about an inequality in [9] not being valid for $\beta = 0$ is misplaced for two reasons. First, the inequality does not appear in [9], but appears near the bottom of the left column of page 494 of [8]. Secondly, in order to prove a finite radius of convergence one needs to prove an inequality with some positive β. Then $\beta = 0$ will be included within the radius of convergence. (It may be noted that there are misprints in [8], probably due to printer errors as Phys. Lett. did not allow authors to correct proofs at that time.)

The next objection in [1] that f is singular at $T = \infty$ is also misleading. The combination $\beta f = -\ln 2$ there, as f has a simple pole at $\beta = 1/k_{\mathrm{B}}T = 0$. One finds that f has a Laurent expansion with pole term $(-\ln 2)/\beta$ followed by a power series in β with a finite radius of convergence. Statistically, at $\beta = 0$ all states have equal probability and there is no phase transition, as not only βf, but also all correlation functions are analytic at $\beta = 0$, in spite of the fact that interactions are turned on once $\beta > 0$.

When Zhang expands $\lambda = Z^{1/N}$ in (A12) and (A13) of [5], he expands $\exp(-\beta f)$, which is equivalent to expanding βf. This makes his objection to expanding βf instead of f unreasonable. The finite radius of convergence proof given in [2] proves that (A13) is not correct.

Response [1] bring up that $1/Z$ has a zero at $\beta = 0$ in the infinite system. But this is again misleading. Yang–Lee theory is only about the zeroes of partition function Z: When zeros pinch the real temperature axis in the large system limit, then there is a phase transition. There is no theorem for $1/Z$.

This pinching of zeros of Z cannot occur, as the proof given in [2] can be extended to the double expansion of βf in β and βB. The proof for the more general cases is in the old literature. From this joint analyticity at $\beta = 0$ and $\beta B = 0$, it follows that zeros are a finite (nonzero) distance away, contradicting any pinching of zeros at $\beta = \beta B = 0$.

3. Other issues

Remark 2.4 in [2] and similar statements show that it is possible to test Zhang's free energy with Monte Carlo methods [10], as one can now estimate both the systematic error due to finite size and the statistical error due to Monte Carlo methods.

Also, the latest various experimental and theoretical estimates for α are significantly different from 0, see the review by Pelissetto and Vicari [11]. One may want to check the accuracies reported of large numbers of theoretical and experimental works that are discussed there and ignored by March and Zhang.

Next, the use of dimensionless $K_i = \beta J_i$ and $h = \beta H$ can be done in more than one way. The partition function and correlation functions (and βf) only depend on these combinations. That some authors set $\beta = 1$, does not mean a loss of the high-temperature case. If one has the result in the K_i and h, one also can choose a new β, say β', and write in the results $K_i = \beta' J_i$ and $h = \beta' H$. There is no loss of the high-T limit $\beta' = 0$. Again this is an objection that is invalid and it does not apply to [2], as I nowhere used $\beta = 1$, nor did I use results from authors that did.

Finally, the last paper [12] is based on an incorrect solution of the 3D Ising model. There are problems I noted: With ϕ a phase, $|e^{i\phi}| = 1$, and formula (4) contains phases that drop out. Also, having three independent Virasoro algebras means that one has the $3+1$ dimensional space rewritten as a 6-dimensional $(2+2+2)$ space, the "product" of three independent 2-dimensional spaces. Things do not add up.

Acknowledgments

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BŁĘDNE ROZWIĄZANIE TRÓJWYMIAROWYCH (3D) PROSTOPADŁOŚCIENNYCH SIATEK ISINGA II

KOMENTARZ DO ODPOWIEDZI NA "BŁĘDNE ROZWIĄZANIE TRÓJWYMIAROWYCH (3D) PROSTOPADŁOŚCIENNYCH SIATEK ISINGA" Z.-D. ZHANGA

S t r e s z c z e n i e

Odpowiedź Zhanga i Marcha na niedawny komentarz o szeregu ich prac jedynie dodaje do poprzednich błędów nowe błędy i mylące stwierdzenia.

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In memory of
Professor Promarz M. Tamrazov

Janusz Garecki

AN INTERESTING PROPERTY OF THE FRIEDMAN UNIVERSES

Summary

We show in the paper that Friedman universes can be created from empty, flat Minkowskian spacetime by using suitable conformal rescaling of the spacetime metric.

Keywords and phrases: universe, Friedman universe, Einstein equations, Cosmological Principle, dust universe, radiation universe

1. Friedman universes

Einstein equations and Cosmological Principle lead us together to Friedman universes. These universes give standard mathematical models of the real Universe.

Einstein equations

$$G_{ik} := R_{ik} - \frac{1}{2}g_{ik}R = \frac{8\pi G}{c^4}T_{ik} =: \beta T_{ik} \tag{1}$$

form system of the ten, 2-nd order quasilinear partial differential equations on ten unknown functions. Solving these equations under given initial and boundary conditions one obtains local geometry of the spacetime, i.e.,

$$g_{ik}(x) \longrightarrow \Gamma^i{}_{kl}(x) \longrightarrow R^i{}_{klm}(x)$$

and local distribution and motion of matter, i.e., $T_{ik}(x)$.

Here G_{ik} is the so-called *Einstein tensor*, T_{ik} is the *matter energy-momentum tensor* (the source of the gravitational field which is represented by tensor G_{ik}), c is the velocity of light in vacuum, and G means Newtonian gravitational constant;

$g_{ik}(x)$ denote components of the metric tensor, and $\Gamma^i{}_{kl}(x)$, $R^i{}_{klm}(x))$ are the Levi-Civita connection and Riemannian curvature components respectively. R_{ik} mean components Ricci tensor and R is the so-called curvature scalar (See, eg., [1]). All Latin indices take values $0, 1, 2, 3$.

The matter tensor $T_{ik}(x)$ consists of $g_{ik},\ u^i,\ p,\ \rho$, where $u^i,\ p,\ \rho$ denote 4-velocity, pressure and density of matter respectively.

Cosmological Principle says that in the largest scale the real Universe is homogeneous and isotropic [1].

In the following we will use *geometrized units* in which $G = c = 1$. Friedman universes are cosmological solutions to the Einstein equations constrained by Cosmological Principle and they are foundation of the relativistic cosmology [1, 2].

The line element

$$ds^2 = g_{ik}(x)dx^i dx^k$$

for these universes, called *Friedman-Lemaitre'-Robertson-Walker* line element, in the *comoving coordinates* $x^0 = t,\ x^1 = \chi,\ \ x^2 = \vartheta,\ \ x^3 = \varphi$, reads

$$ds^2 = dt^2 - R^2(t)\big[d\chi^2 + S^2(\chi)\big(d\vartheta^2 + sin^2\vartheta d\varphi^2\big)\big], \tag{2}$$

where

$$\begin{aligned} S(\chi) &= sin\chi, \quad if \quad k = 1 \\ S\chi &= \chi, \quad if \quad k = 0 \\ S(\chi) &= sh\chi, \quad if \quad k = (-)1. \end{aligned} \tag{3}$$

t is the *cosmic time*, i.e., the proper time for *isotropic observers*, which are at rest in the coordinates $(t, \chi, \vartheta, \varphi)$.

An isotropic observer O represents center of mass of a cluster of galaxies in real Universe. $R(t)$ is the so-called *scale factor* (it scales spatial distances) and $k = 0, \pm 1$ means the *normalized curvature* (curvature index) of the spatial sections $x^0 = t = const$.

If $k = 1$, then we have closed (spherical or elliptical) spatial sections, if $k = 0$ the geometry of the spatial section is flat, and if $k = (-)1$, then the geometry of spatial sections is hyperbolic.

Usually one chooses the moment $t = 0$ of the cosmic time t when $R = 0$, i.e., usually one has $R(0) = 0$.

Einstein equations with perfect fluid (incompressible fluid, without any viscosity and not conducting heat) as source [2] reduce, for the FLRW line element (2)–(3) to the *Friedman equations*

$$\frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} = \frac{\rho}{2\beta}, \tag{4}$$

[1] We are modelling *cosmological substrat* by using an ideal (or perfect) fluid.

[2] A particle of this fluid represents a cluster of galaxies in real Universe.

$$\frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R} + \frac{k}{R^2} = (-)\frac{p}{2\beta}. \tag{5}$$

Here $\beta = 8\pi$ (We use geometrized units), $\rho = \rho(t)$ means the rest density of the fluid, and $p = p(\rho) = p(t)$ — its pressure. $\dot{R} := \frac{dR}{dt}$, and $\ddot{R} := \frac{d^2R}{dt^2}$.

Caloric equation $p = p(\rho)$ must be added to Friedman equations (4)-(5) in order to get a determined system on the three unknown functions: $R = R(t)$, $\rho = \rho(t)$, $p = p(t)$.

Usually one considers solutions to the Friedman equations (4)–(5) in the two extreme cases: $p = 0$ (dust universes or matter dominant universes, in short **MDU**), and $p = \frac{\rho}{3}$ (radiation dominant universes, in short **RDU**).

We will confine to solutions in these two extreme cases.

Dust universes (**MDU**) with $p = 0$:

1. $k = 1$ (closed universe). In this case we have parametric solution

$$\begin{aligned} R &= M\big(1 - \cos\eta\big), \\ t &= M\big(\eta - \sin\eta\big). \end{aligned} \tag{6}$$

$0 < \eta < 2\pi$.

2. $k = 0$ (flat universe). In this case

$$R = \Big(\frac{9M}{2}t^2\Big)^{1/3}, \quad 0 < t < \infty. \tag{7}$$

3. $k = (-)1$ (open universe). In the case we also have parametric solution

$$\begin{aligned} R &= M\big(\cosh\eta - 1\big), \\ t &= M\big(\sinh\eta - \eta\big), \quad 0 < \eta < \infty. \end{aligned} \tag{8}$$

Here η denotes a parameter and $M = (4/3)\pi R^3\rho$ is the first integral of the Friedman equations. Physically M is the mass contained inside of a “sphere” having volume $(4/3)\pi R^3$.

Radiation universes (**RDU**) with $p = \frac{\rho}{3}$

1. $k = 1$ (closed universe)

$$R = \sqrt{(2bt - t^2)}, \quad b := \sqrt{\frac{8\pi C}{3}}, \quad 0 < t < 2b, \tag{9}$$

where $C = \rho R^4 = const > 0$ is the first integral of the Friedman equations in this case.

2. $k = 0$ (flat universe)

$$R = \sqrt{2bt}, \quad 0 < t < \infty. \tag{10}$$

3. $k = (-)1$ (open universe)

$$R = \sqrt{(2bt + t^2)}, \quad 0 < t < \infty. \tag{11}$$

Having $R = R(t)$ one can find $\rho(t)$ from the first integrals and then $p = p(t)$ from caloric equations.

It is believed that one of the **MDU** correctly describes present stage of the Universe, and that one of the **RDU** correctly describes early Universe[3].

It is seen from (6)–(11) that the Friedman universes are singular at least in one moment of the cosmic time t (In this moment $R = 0$). These singularities are inevitable in classical general relativity (Theorems by Hawking and Penrose, and Senovilla [3]); but "quantized general relativity" (loops quantum gravity) seems remove these singularities (Ashtekar, Bojowald and Lewandowski) [4].

2. Conformal rescaling of metric and conformally flat spacetimes

By *conformal rescaling* of the metric g we mean the following transformation (in established coordinates)

$$\hat{g}_{ab}(x) = \Omega^2(x) g_{ab}(x), \tag{12}$$

where the *conformal factor* $\Omega(x)$ is dimensionless, smooth and positive.

One can immediately get from (12) that

$$\hat{g}^{ab}(x) = \Omega^{(-)2}(x) g^{ab}(x), \tag{13}$$

and, after some tedious calculations one can obtain other useful transformational formulas [5]. For our future aims the following formulas will be needed

$$\hat{R}^b{}_d = \Omega^{(-)2} R^b{}_d + 2\Omega^{(-)1}\big(\Omega^{(-)1}\big)_{;dc} g^{bc} - \frac{1}{2}\Omega^{(-)4}\big(\Omega^2\big)_{;ac} g^{ac}\delta^b_d, \tag{14}$$

$$\hat{R} = \Omega^{(-)2} R - 6\Omega^{(-)3}\Omega_{;cd} g^{cd}, \tag{15}$$

and

$$\hat{T}_i{}^k = \Omega^{(-)4} T_i{}^k. \tag{16}$$

Here $;a$ is covariant derivative with respect Levi-Civita connection of the metric in the initial gauge $g_{ab}(x)$.

A spacetime is *conformally flat* if there exist holonomic coordinates ($x^0 = t,\ x^1 = x,\ x^2 = y,\ x^3 = z$) in which its line element ds^2 has the form

$$\begin{aligned} ds^2 &= \Omega^2(x^0, x^1, x^2, x^3)\big(dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2}\big) \\ &\equiv \Omega^2(x^0, x^1, x^2, x^3)\eta_{ik} dx^i dx^k. \end{aligned} \tag{17}$$

The

$$\eta_{ik} dx^i dx^k = dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2} \tag{18}$$

means the line element of the empty, flat Minkowski spacetime in inertial coordinates.

[3]The recent large–scale astronomical observations seem favorize an accelerated flat model.

The Theorem is true:

Necessary and sufficient condition of the conformal flateness of the 4-dimensional (or higher, $n > 4$ dimensional) spacetime is vanishing its *Weyl tensor* C_{abcd}, where

$$C_{abcd} := R_{abcd} + \frac{R(g_{ac}g_{bd} - g_{ad}g_{bc})}{(n-1)(n-2)} - \tag{19}$$
$$- \frac{\big(g_{ac}R_{bd} - g_{bc}R_{ad} + g_{bd}R_{ac} - g_{ad}R_{bc}\big)}{(n-2)}.$$

In the above formula R_{abcd} are components of the Riemann tensor, R_{ab} denote Ricci tensor components and R means Riemannian curvature scalar.

In the framework of general relativity Weyl's tensor C_{abcd} describes free gravitational field (tidal forces).

An example of the conformally flat spacetimes give Friedman universes.

3. Conformal transformation as Creator of the Friedman universes

We have under conformal rescaling of the metric (12) if we use the formulas (14)-(15)

$$\begin{aligned} \hat{G}^d_b &= \hat{R}^d_b - \frac{1}{2}\delta^d_b \hat{R} = \Omega^{(-)2} G^d_b \\ &+ \frac{2}{\Omega}\big(\Omega^{(-)1}\big)_{;bc} g^{dc} + \frac{3}{\Omega^3}\delta^d_b \Omega_{;ce} g^{ce} \\ &- \frac{1}{2\Omega^4}\big(\Omega^2\big)_{;ac} g^{ac}\delta^d_b; \end{aligned} \tag{20}$$

$$\hat{T}^d_b = \Omega^{(-)4} T^d_b. \tag{21}$$

By using Einstein equations in *old gauge* $g_{ik}(x)$

$$G^d_b = \beta T^d_b \tag{22}$$

one can combine (19)-(20) to the form

$$\hat{G}^d_b = \beta\Omega^2 \hat{T}^d_b + \beta \widetilde{T}^d_b, \tag{23}$$

where

$$\begin{aligned} \widetilde{T}^d_b &:= \frac{1}{\beta}\Big[\frac{2}{\Omega}\big(\Omega^{(-)1}\big)_{;bc} g^{dc} \\ &+ \frac{\delta^d_b}{\Omega^3}\big(3\Omega_{;ce} g^{ce} - \frac{\Omega^2{}_{;ac}}{2\Omega} g^{ac}\big)\Big]. \end{aligned} \tag{24}$$

(22) gives Einstein equations in *new gauge* $\hat{g}_{ik}(x)$.

The tensor $\widetilde{T}^d_b(x)$ is the energy-momentum tensor of this matter *which was created* by conformal rescaling of the initial metric $g_{ik}(x)$ while the tensor $\hat{T}^d_b(x)$ is transformed , following (20), the matter tensor $T^d_b(x)$ which have already existed in the *old gauge* $g_{ik}(x)$.

One can rewrite (22) to the form

$$\hat{G}^d_b = \beta \bar{T}^d_b, \tag{25}$$

where

$$\bar{T}^d_b := \Omega^2 \hat{T}^d_b + \widetilde{T}^d_b. \tag{26}$$

Friedman universes are conformally flat. So, we can take in the case as "initial conditions"

$$g_{ik}(x) = \eta_{ik}, \quad G^d_b = 0, \quad T^d_b = 0 \longrightarrow \hat{T}^d_b(x) = 0, \tag{27}$$

i.e., we can take empty Minkowskian spacetime as initial spacetime. Doing so, one can get the metric tensor of a Friedman universe in the form

$$\hat{g}_{ik}(x) = \Omega^2(x)\eta_{ik}, \tag{28}$$

where conformal factor $\Omega(x)$ depends on Friedman universe.

Thus, metric $\hat{g}_{ik}(x)$ of a Friedman universe, i.e., *whole geometry* of a Friedman universe can be obtained from empty Minkowskian spacetime by a suitable conformal rescaling of the Minkowskian metric. Material content of this universe can be easily obtained from Einstein equations

$$\widetilde{T}^d_b := \frac{1}{\beta}\hat{G}^d_b, \tag{29}$$

where $\hat{G}^d_b(x)$ is Einstein tensor calculated from $\hat{g}_{ik}(x)$.

As an example we will consider a flat Friedman universe.

In this case

$$\hat{g}_{ik}(x) = \Omega^2(\tau)\eta_{ik} = \Omega^2(\tau)(d\tau^2 - dx^2 - dy^2 - dz^2) \tag{30}$$

with $\Omega(\tau) \equiv R(\tau)$. τ is here the so-called *conformal time* [6].

After a simple but tedious calculation one gets from (28) that

$$\begin{aligned} \widetilde{T}_0{}^0 &= \frac{3R'}{\beta R^4} \;\; (= \rho) \\ \widetilde{T}_1{}^1 &= \widetilde{T}_2{}^2 = \widetilde{T}_3{}^3 = \frac{1}{\beta R^3}\big(2R'' - \frac{R'^2}{R}\big) \;\; (= -p). \end{aligned} \tag{31}$$

Here prime denotes derivation with respect conformal time τ.

Other components of the energy-momentum tensor $\widetilde{T}_b{}^a$ of the matter created by conformal rescaling (29) of the Minkowskian metric are vanishing.

For the flat dust Friedman universe we obtain

$$ds^2 = R^2(\tau)\big(d\tau^2 - dx^2 - dy^2 - dz^2\big), \tag{32}$$

where

$$R(\tau) = \frac{A^3}{9}\tau^2, \quad A = \big(6\pi\rho R^3\big)^{1/3} = \text{const}. \tag{33}$$

From that one gets

$$R' = \frac{2A^3\tau}{9}, \quad R'' = \frac{2A^3}{9}, R''' = 0, \tag{34}$$

and higher derivatives also vanish.

In consequence, the material content of the universe following (30) reads

$$\widetilde{T}_0{}^0 = \rho = \frac{972}{\beta A^6 \tau^6}. \tag{35}$$

The other components of the tensor $\widetilde{T}_a{}^b$ are vanishing, i.e., $p = 0$ and stresses vanish (as it should be in the case).

Thus, we have correctly created flat, dust Friedman universe from empty Minkowskian spacetime by using the conformal transformation (31)–(32).

4. Conclusion

As we could see, Friedman universes *can be created* by a suitable conformal rescaling of the flat Minkowskian metric,i.e., these universes *can be created* from empty, flat Minkowskian spacetime by conformal transformations.

Therefore, we needn't any "quantum gravity" in order to explain origin of the Friedman universes: classical conformal transformations are sufficient.

The analogical statement is, of course, correct for any other conformally flat spacetime.

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INTERESUJĄCA WŁASNOŚĆ MODELI KOSMOLOGICZNYCH FRIEDMANA

S t r e s z c z e n i e

W tej pracy pokazano, że modele kosmologiczne Friedmana, które są podstawą współczesnej kosmologii, można wykreować z pustej czasoprzestrzeni Minkowskiego przy pomocy odpowiedniej transformacji konforemnej.

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B U L L E T I N

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In memory of
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VECTORS OF A SQUARE MATRIX IN $\mathbb{R}^n$

Summary

The purpose of this paper is to introduce that an arbitrary non-singular n-th order square matrix can be represented as a matrix associated with an arbitrary versor lying in the n-dimensional Cartesian space in such a way that the successive terms of this matrix are the products of the coordinates of this versor and the corresponding n coordinates of different vectors lying in the same space. We next demonstrated that if the versor is one of the eigenversors of matrix, then the direction of it determines the distance, while the eigenvalue is equal to the distance of the plane defined by these vectors from the origin. Finally, the paper presents selected examples for the matrix of the second and third row.

Keywords and phrases: external product of two vectors, matrix eigenvalue calculus

1. Dyad in the multi product of vectors

A dyad or, in other words, *the external product of two vectors*, is a rectangle $n \times m$ dimensional matrix, of the form $\boldsymbol{P}_{ab}$, containing products of the coordinates of two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ (in matrix form $\boldsymbol{a} = [a_1\ a_2\ \dots\ a_n]^T$ and $\boldsymbol{b} = [b_1\ b_2\ \dots\ b_m]^T$), in a general case, of the vectors defined in different dimensional spaces $n \neq m$. The terms of a dyad are products of the coordinates of vectors equal to following equation, respectively:

$$\boldsymbol{P}_{ab} = [a_i b_j] \quad (i = 1,\ 2,\ \dots,\ n;\ \ j = 1,\ 2,\ \dots,\ m). \tag{1}$$

The matrix (1) corresponds to the product of two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ determined in the form of different dimensional matrices and at the same time to the product written in such a form (in such an order) that the multiplication of the matrix is impossible. If

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} \quad \text{and} \quad \boldsymbol{b}^T = [b_1 \ \ b_2 \ \ \dots \ \ b_m]^T, \tag{2}$$

then the notation of the product of the matrix

$$\boldsymbol{a}\boldsymbol{b}^T = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} [b_1 \ \ b_2 \ \ \dots \ \ b_m]^T \tag{3}$$

is an equivalent of the dyad $\boldsymbol{P}_{ab}$ in the multi-product of vectors. The matrix $\boldsymbol{P}_{ab}$ can be used for the notation of multi-products of vectors of any order, using any pairs of scalar products.

Thus, the dyad $\boldsymbol{P}_{ab}$, which is a square matrix ($n = m$), is matrix of product of two vectors in case when two vectors lie in the same space. It has then the following form:

$$\boldsymbol{P}_{ab} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_n \\ \dots & \dots & \ddots & \dots \\ a_n b_1 & a_n b_2 & \dots & a_n b_n \end{bmatrix}. \tag{4}$$

It is easy to calculate that the matrix determinant of two vectors product $\boldsymbol{P}_{ab}$ (for $n = m$) is equal to zero ($\det \boldsymbol{P}_{ab} = 0$). Thus, dyad (4) is singular matrix.

A good illustration of the possibilities offered by the use of a dyad in the notation of arbitrary multi-products are the simplest odd and even products which – after introducing – can be formed identically in many ways; at the same time, the vectors can be reordered in different possible ways in scalar products. The correctness of the cited identities can be checked by performing appropriate transformations.

For example, a product of the third order $(\vec{\boldsymbol{a}}\vec{\boldsymbol{b}})\vec{\boldsymbol{c}}$ can be written in four different ways, using four following dyads:

$$\begin{aligned} (\boldsymbol{a}^T\boldsymbol{b})\boldsymbol{c}^T &= \boldsymbol{a}^T\boldsymbol{P}_{bc} \quad \text{or} \quad (\boldsymbol{b}^T\boldsymbol{a})\boldsymbol{c}^T = \boldsymbol{b}^T\boldsymbol{P}_{ac}, \\ \boldsymbol{c}(\boldsymbol{a}^T\boldsymbol{b}) &= \boldsymbol{P}_{ca}\boldsymbol{b} \qquad \text{or} \quad \boldsymbol{c}(\boldsymbol{b}^T\boldsymbol{a}) = \boldsymbol{P}_{cb}\boldsymbol{a}. \end{aligned} \tag{5}$$

A special type of multi-product of three vectors is the one, which includes versors of an arbitrary axis (an arbitrary vector). For example, the versor $\vec{\boldsymbol{e}}_c$, such that the matrix of its coordinates has the form $\boldsymbol{e}_c = [c_1 \ c_2 \ ... \ c_n]^T$. Then a vector projection of $\vec{\boldsymbol{b}}_c$ the arbitrary vector $\vec{\boldsymbol{b}}$ in the direction of the versor $\vec{\boldsymbol{e}}_c$ can be written, in the classic and matrix form respectively.

$$\vec{\boldsymbol{b}}_c = (\vec{\boldsymbol{b}} \ \vec{\boldsymbol{e}}_c) \ \vec{\boldsymbol{e}}_c \quad \text{or} \quad \boldsymbol{e}_c(\boldsymbol{b}^T\boldsymbol{e}_c) = \boldsymbol{P}_{e_c b} \ \boldsymbol{e}_c, \tag{6}$$

where

$$(7) \qquad \boldsymbol{P}_{e_c b} = \begin{bmatrix} c_1 b_1 & c_1 b_2 & \dots & c_1 b_n \\ c_2 b_1 & c_2 b_2 & \dots & c_2 b_n \\ \dots & \dots & \ddots & \dots \\ c_n b_1 & c_n b_2 & \dots & c_n b_n \end{bmatrix}.$$

The matrix $\boldsymbol{P}_{e_c b}$ here is the transformation matrix of vector $\vec{\boldsymbol{b}}$ in the direction of the versor $\vec{\boldsymbol{e}}_c$.

2. Eigenvector and eigenvalue of any matrix

2.1.

An arbitrary n-th order square matrix $\boldsymbol{A}$,

$$(8) \qquad \boldsymbol{A} = [a_{ij}] \qquad (i,\ j = 1,\ 2, \dots,\ n).$$

with elements a_{ij} that are real numbers, whose determinant det $\boldsymbol{A} \neq 0$,

$$(9) \qquad \boldsymbol{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

can be written as a matrix associated with any versor $\vec{\boldsymbol{e}}_c$, which lies in the space $\mathbb{R}^{\boldsymbol{n}}$, such that the coordinates c_i in this space form a matrix

$$(10) \qquad \boldsymbol{e}_c = [\vec{\boldsymbol{e}}_c\vec{\boldsymbol{e}}_1 \ \ \vec{\boldsymbol{e}}_c\vec{\boldsymbol{e}}_2 \ \dots \ \vec{\boldsymbol{e}}_c\vec{\boldsymbol{e}}_n]^T = [c_1 \ \ c_2 \ \dots \ c_n]^T$$

and the elements of this matrix satisfy the condition

$$(11) \qquad \sum_{i=1}^{n} c_i^2 = 1.$$

The relation of elements of matrix $\boldsymbol{A}$ with the coordinates of any versor $\vec{\boldsymbol{e}}_c$ obtained in such a way assuming that subsequent elements a_{ij} of matrix $\boldsymbol{A}$ are the products of the coordinate c_i of versor $\vec{\boldsymbol{e}}_c$ and the corresponding numbers b_{ij}, such that $a_{ij} = c_i b_{ij}$. Then

$$(12) \qquad \boldsymbol{A} = [a_{ij}] = [c_i b_{ij}] = \begin{bmatrix} c_1 b_{11} & c_1 b_{12} & \dots & c_1 b_{1n} \\ c_2 b_{21} & c_2 b_{22} & \dots & c_2 b_{2n} \\ \dots & \dots & \ddots & \dots \\ c_n b_{n1} & c_n b_{n2} & \dots & c_n b_{nn} \end{bmatrix}.$$

Thus, the matrix $\boldsymbol{A}$, by analogy to the dyad of two vectors (4), (6), can be written as a matrix $\boldsymbol{A}_{e_c b_i}$ and then

$$\boldsymbol{A} = \boldsymbol{A}_{e_c b_i} = \begin{bmatrix} c_1 b_{11} & c_1 b_{12} & \dots & c_1 b_{1n} \\ c_2 b_{21} & c_2 b_{22} & \dots & c_2 b_{2n} \\ \dots & \dots & \ddots & \dots \\ c_n b_{n1} & c_n b_{n2} & \dots & c_n b_{nn} \end{bmatrix}. \tag{13}$$

From the form of matrix (13) one can conclude that the elements b_{ij} in successive rows of the matrix $\boldsymbol{A}_{e_c b_i}$ are, in the $\mathbb{R}^{\boldsymbol{n}}$ space, the j-th coordinates of certain vectors $\vec{\boldsymbol{b}}_i$. These coordinates form the matrices

$$\boldsymbol{b}_i = [b_{i1}\ b_{i2}\ \dots\ b_{in}]^T. \tag{14}$$

The product of such a matrix $\boldsymbol{A}_{e_c b_i}$ - treated as a matrix associated with the versor $\vec{\boldsymbol{e}}_c$, and a matrix of coordinates of the versor $\vec{\boldsymbol{e}}_c$ is a column matrix $\boldsymbol{a}_{\beta_i}$ of the form

$$\begin{aligned} \boldsymbol{a}_{\beta_i} = \boldsymbol{A}_{e_c b_i} \boldsymbol{e}_c &= \begin{bmatrix} c_1 b_{11} & c_1 b_{12} & \dots & c_1 b_{1n} \\ c_2 b_{21} & c_2 b_{22} & \dots & c_2 b_{2n} \\ \dots & \dots & \ddots & \dots \\ c_n b_{n1} & c_n b_{n2} & \dots & c_n b_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{bmatrix} \\ &= \begin{bmatrix} (\boldsymbol{b}_1 \vec{\boldsymbol{e}}_c) c_1 \\ (\boldsymbol{b}_2 \vec{\boldsymbol{e}}_c) c_2 \\ \dots \\ (\boldsymbol{b}_n \vec{\boldsymbol{e}}_c) c_n \end{bmatrix} = \begin{bmatrix} \beta_1 c_1 \\ \beta_2 c_2 \\ \dots \\ \beta_n c_n \end{bmatrix}. \end{aligned} \tag{15}$$

Scalar product of successive vectors $\vec{\boldsymbol{b}}_i$ and vector $\vec{\boldsymbol{e}}_c$, occurring in the coordinates of the matrix (15)

$$\beta_i = \vec{\boldsymbol{b}}_i \vec{\boldsymbol{e}}_c \tag{16}$$

are equal to the lengths of projections of vectors $\vec{\boldsymbol{b}}_i$ onto the direction of the versor $\vec{\boldsymbol{e}}_c$. Such a matrix is the so-called transformation matrix of successive vectors $\vec{\boldsymbol{b}}_i$, written in subsequent rows of this matrix, to common direction of versor $\vec{\boldsymbol{e}}_c$ and also their assignment to successive coordinates of newly established vector.

Following a reasoning presented above, one can conclude that every square n-th order matrix with real words, multiplied by coordinates matrix of an arbitrary n-th dimensional versor gives a vector, whose successive coordinates are the products of appropriate coordinates of the selected versor and the length of projection of certain, successive n vectors $\vec{\boldsymbol{b}}_i$ onto the direction of this versor. Depending on the position of versor $\vec{\boldsymbol{e}}_c$ in space $\mathbb{R}^{\boldsymbol{n}}$ its coordinates and the coordinates of successive vectors $\vec{\boldsymbol{b}}_i$ charge, and so do successive scalar products β_i (16).

Let us demand that the versor $\vec{\boldsymbol{e}}_c$ obtains such a direction in space $\mathbb{R}^{\boldsymbol{n}}$, for which all scalar products β_i have the same value, which means that lengths of all projections of successive vectors $\vec{\boldsymbol{b}}_i$ are equal. Then

$$\forall_{i,j}\ \vec{\boldsymbol{b}}_i \vec{\boldsymbol{e}}_c = \vec{\boldsymbol{b}}_j \vec{\boldsymbol{e}}_c, \quad \text{and thus} \quad \forall_{i,j}\ \beta_i = \beta_j = \alpha. \tag{17}$$

Thus, defined versor $\vec{\boldsymbol{e}}_c$ is described in literature as the eigenvector of the matrix $\boldsymbol{A}$, while the common value of dot products, designated here as α, is called an eigenvalue of this matrix. Assuming (17) an already known relation was obtained

$$\boldsymbol{A}\boldsymbol{e}_c = \alpha\boldsymbol{e}_c \tag{18}$$

in the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{bmatrix} = \alpha \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{bmatrix}. \tag{19}$$

Similar equation is obtained for the second form of a matrix $\boldsymbol{A} = \boldsymbol{A}_{e_c b_i}$:

$$\boldsymbol{A}_{e_c b_i}\ \boldsymbol{e}_c = \alpha\boldsymbol{e}_c \ \text{ or } \begin{bmatrix} c_1 b_{11} & c_1 b_{12} & \dots & c_1 b_{1n} \\ c_2 b_{21} & c_2 b_{22} & \dots & c_2 b_{2n} \\ \dots & \dots & \ddots & \dots \\ c_n b_{n1} & c_n b_{n2} & \dots & c_n b_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{bmatrix} = \alpha \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{bmatrix}. \tag{20}$$

In order to determine the eigenvalue and eigenvector, the equation (13) is written in following form

$$\boldsymbol{A}\boldsymbol{e}_c - \alpha\boldsymbol{e}_c = 0, \quad \boldsymbol{A}\boldsymbol{e}_c - \alpha\boldsymbol{I}_n\boldsymbol{e}_c = 0, \tag{21}$$

where $\boldsymbol{I}_n$ is the n-th order identity matrix. Then the equation is converted and written with the use of a new matrix $\boldsymbol{A}_\alpha$:

$$\boldsymbol{A}_\alpha\boldsymbol{e}_c = 0, \tag{22}$$

that is a form

$$\boldsymbol{A}_\alpha = \begin{bmatrix} a_{11}-\alpha & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\alpha & \dots & a_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\alpha \end{bmatrix}. \tag{23}$$

The condition for the existence of a solution of the equation (22) is that the determinant of matrix $\boldsymbol{A}_\alpha$ is equal to zero. After expanding the determinant $\det \boldsymbol{A}_\alpha$, the characteristic equation $\det \boldsymbol{A}_\alpha = 0$ is obtained corresponding to the matrix (23).

Roots $\alpha_i \quad (i = 1,\ 2, \dots,\ k;\ k \le n)$ of the characteristic equation are the eigenvalues of the matrix $\boldsymbol{A}$. Every eigenvalue corresponds to a different eigenvector $\vec{\boldsymbol{e}}_c$, whose matrix of coordinates $\boldsymbol{e}_c = [c_1\ c_2\ \dots\ c_n]^T$ can be determined from the equation (23) with the condition (11). Using the matrix $\boldsymbol{A}$ (12)(13) and knowing the coordinates of the eigenvector, the coordinates matrix $\boldsymbol{b}_i = [b_{i1}\ b_{i2}\ \dots\ b_{in}]^T$ of successive vectors $\vec{\boldsymbol{b}}_i$ can be calculated.

2.2.

Vectors $\vec{\boldsymbol{b}}_i$, after applying them in the origin of an orthogonal n-th dimensional system of coordinates $0x_1 \ \dots \ x_n$, specify in this system the location n of points $B_i = [b_{i1} \ b_{i2} \ \dots \ b_{in}]$. These points B_i span the $n-1$ dimensional hyperplane whose equation

$$a_0 + a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_{n-1}x_{n-1} + a_nx_n = 0 \tag{24}$$

is obtained from series expansion of $n+1$-th order determinant having form

$$\begin{bmatrix} 1 & x_1 & x_2 & \dots & x_n \\ 1 & b_{11} & b_{12} & \dots & b_{1n} \\ 1 & b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \ddots & \dots \\ 1 & b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} = 0. \tag{25}$$

Determinants $a_i \quad (i \ = \ 0, \ 1, \ 2, \ \dots, \ n)$ of hyperplane equation are the minors' values of the n-th order determinant (25). These minors are the algebraic complements of successive terms x_i in the expansion of the determinant (25) by the first row. Hyperplane (24) satisfies the condition of orthogonal to the eigenvector $\vec{\boldsymbol{e}}_c$ of matrix $\boldsymbol{A}$ (9)(12).

Arbitrary vector $\vec{\boldsymbol{b}}_i$ ($\boldsymbol{b}_i = [b_{i1} \ b_{i2} \ \dots \ b_{in}]^T$) can be projected onto two directions: the direction of the eigenversor $\vec{\boldsymbol{e}}_c$ of matrix $\boldsymbol{A}$ as a vector $\vec{\boldsymbol{b}}_{ic}$ and the direction orthogonal to eigenversor, lying on the hyperplane π orthogonal to the versor as a vector $\vec{\boldsymbol{b}}_{i\pi}$.

This gives the vector $\vec{\boldsymbol{b}}_i = \vec{\boldsymbol{b}}_{ic} + \vec{\boldsymbol{b}}_{i\pi}$ (in the matrix form $\boldsymbol{b}_i = \boldsymbol{b}_{ic} + \boldsymbol{b}_{i\pi}$), such that the projection of this vector $\vec{\boldsymbol{b}}_i$ onto the direction of eigenversor is a vector

$$\vec{\boldsymbol{b}}_{ic} = \alpha \vec{\boldsymbol{e}}_c \quad \text{of coordinates matrix} \quad \boldsymbol{b}_{ic} = \alpha \boldsymbol{e}_c \tag{26}$$

and $\alpha = \beta_i = (\vec{\boldsymbol{b}}_i \vec{\boldsymbol{e}}_c)$ (16) (17), so $\vec{\boldsymbol{b}}_{ic} = (\vec{\boldsymbol{b}}_i \vec{\boldsymbol{e}}_c)\vec{\boldsymbol{e}}_c$. Following that, the matrix coordinates $\boldsymbol{b}_{ic}$ vector $\vec{\boldsymbol{b}}_{ic}$ is obtained as the product of the dyad $\boldsymbol{P}_{e_c e_c}$ (4), (6) and matrix of coordinates $\boldsymbol{b}_i$ vector $\vec{\boldsymbol{b}}_i$,

$$\boldsymbol{b}_{ic} = \boldsymbol{P}_{e_c e_c} \boldsymbol{b}_i, \qquad \boldsymbol{P}_{e_c e_c} = \begin{bmatrix} c_1c_1 & c_1c_2 & \dots & c_1c_n \\ c_2c_1 & c_2c_2 & \dots & c_2c_n \\ \dots & \dots & \ddots & \dots \\ c_nc_1 & c_nc_2 & \dots & c_nc_n \end{bmatrix}. \tag{27}$$

The projection vector $\vec{\boldsymbol{b}}_{i\pi} = \vec{\boldsymbol{b}}_i - \vec{\boldsymbol{b}}_{ic} = \vec{\boldsymbol{b}}_i - \alpha \vec{\boldsymbol{e}}_c$ onto the hyperplane π is in the matrix form

$$\boldsymbol{b}_{i\pi} = \boldsymbol{b}_i - \alpha \boldsymbol{e}_c \quad \text{or} \quad \boldsymbol{b}_{i\pi} = \boldsymbol{b}_i - \boldsymbol{P}_{e_c e_c} \boldsymbol{b}_i = \boldsymbol{I}_n \boldsymbol{b}_i - \boldsymbol{P}_{e_c e_c} \boldsymbol{b}_i. \tag{28}$$

Then

$$(29) \qquad \boldsymbol{b}_{i\pi} = \begin{bmatrix} 1-c_1c_1 & c_1c_2 & \dots & c_1c_n \\ c_2c_1 & 1-c_2c_2 & \dots & c_2c_n \\ \dots & \dots & \ddots & \dots \\ c_nc_1 & c_nc_2 & \dots & 1-c_nc_n \end{bmatrix} \begin{bmatrix} b_{i1} \\ b_{i2} \\ \dots \\ b_{in} \end{bmatrix}.$$

The vector sum of all vectors $\vec{\boldsymbol{b}}_i$,

$$(30) \qquad \boldsymbol{b}_i = [b_{i1}\ b_{i2}\ \dots\ b_{in}]^T,$$

appearing in subsequent matrix rows $\boldsymbol{A} = \boldsymbol{A}_{e_c b_i}$ (13), is vector $\vec{\boldsymbol{b}}$ such that the matrix of coordinates has the form

$$(31) \qquad \boldsymbol{b} = \begin{bmatrix} b_1 = \sum b_{i1} & b_2 = \sum b_{i2} & \dots & b_n = \sum b_{in} \end{bmatrix}^T.$$

The projection $\vec{\boldsymbol{b}}_c$ of the vector $\vec{\boldsymbol{b}}$ onto the direction eigenversor $\vec{\boldsymbol{e}}_c$ is the n-fold projection of projection vector $\vec{\boldsymbol{b}}_{ic}$ of each vector $\vec{\boldsymbol{b}}_i$ onto this direction. Therefore

$$(32) \qquad \vec{\boldsymbol{b}}_c = n\alpha\vec{\boldsymbol{e}}_c \quad \text{and} \quad \boldsymbol{b}_c = n\alpha\boldsymbol{e}_c \quad \text{or} \quad \boldsymbol{b}_c = n\boldsymbol{P}_{e_ce_c}\boldsymbol{b}_i.$$

On the other hand, vector of projection $\vec{\boldsymbol{b}}_\pi$ of vector $\vec{\boldsymbol{b}}$ onto the plane π, $\vec{\boldsymbol{b}}_\pi = \vec{\boldsymbol{b}} - \vec{\boldsymbol{b}}_c = \vec{\boldsymbol{b}} - n\alpha\vec{\boldsymbol{e}}_c$, has coordinates that can be written in matrix form

$$(33) \qquad \boldsymbol{b}_\pi = \boldsymbol{b} - n\alpha\boldsymbol{e}_c \quad \text{or} \quad \boldsymbol{b}_\pi = \boldsymbol{b} - \boldsymbol{P}_{e_ce_c}\boldsymbol{b} = \boldsymbol{I}_n\boldsymbol{b} - \boldsymbol{P}_{e_ce_c}\boldsymbol{b}.$$

Then

$$(34) \qquad \boldsymbol{b}_{\pi} = \begin{bmatrix} 1-c_1c_1 & c_1c_2 & \dots & c_1c_n \\ c_2c_1 & 1-c_2c_2 & \dots & c_2c_n \\ \dots & \dots & \ddots & \dots \\ c_nc_1 & c_nc_2 & \dots & 1-c_nc_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}.$$

Assumed here reference system in the space $\mathbb{R}^{\boldsymbol{n}}$, consisting of axes overlapping with the eigenvector of matrix $\boldsymbol{A}$ (9) and orthogonal to this axis $n-1$-dimensional hyperplane π can be called the umbrella associated with the matrix $\boldsymbol{A}$. One shall remember, that the number of umbrellas associated with the matrix $\boldsymbol{A}$ is as many as the number of the eigenvectors of that matrix, which means at most n.

3. An example in $\mathbb{R}^2$

Assumed a square matrix with the following elements

$$(35) \qquad \boldsymbol{A} = \begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix}, \quad \det A = -15.$$

We use the form of equation (22).

The existence condition of solutions of equation (22) has the form $\det \boldsymbol{A}_\alpha = 0$ and its series expansions gives the characteristic equation

$$\det \boldsymbol{A}_\alpha = \det \begin{bmatrix} -1-\alpha & 6 \\ 2 & 3-\alpha \end{bmatrix} = 0 \quad \longmapsto \quad \alpha^2 - 2\alpha - 15 = 0. \tag{36}$$

This yields to a conclusion that matrix $\boldsymbol{A}$ has two eigenvalues $\alpha = -3$ and $\alpha = 5$.

3.1. Eigenvalue $\alpha = -3$

From equation (22) after substituting for the matrix (23) for $\alpha = -3$ one can obtain

$$\begin{bmatrix} 2 & 6 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0. \tag{37}$$

Hence and from the condition (11) is obtained system of equations:

$$\begin{cases} c_1 + 3c_2 = 0 \\ c_1^2 + c_2^2 = 1 \end{cases}. \tag{38}$$

Two pairs of numbers, which are coordinates of eigenvectors $\boldsymbol{e}_c = [c_1\ c_2]^T$ of matrix $\boldsymbol{A}$, are solution of that system of equations in the form

$$\boldsymbol{e}_c = \begin{bmatrix} \frac{3\sqrt{10}}{10} & -\frac{\sqrt{10}}{10} \end{bmatrix}^T \quad \text{and} \quad \boldsymbol{e}_c = \begin{bmatrix} -\frac{3\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \end{bmatrix}^T. \tag{39}$$

Comparing values of the matrix's elements (6) and (13) one obtains coordinates of vectors $\vec{\boldsymbol{b}}_1$ and $\vec{\boldsymbol{b}}_2$ in the matrix form of their coordinates for both eigenvectors

$$\begin{aligned}
&\text{for} \quad \boldsymbol{e}_c = \begin{bmatrix} \frac{3\sqrt{10}}{10} & -\frac{\sqrt{10}}{10} \end{bmatrix}^T: \\
&\qquad \boldsymbol{b}_1 = \begin{bmatrix} -\frac{\sqrt{10}}{3} & 2\sqrt{10} \end{bmatrix}^T \quad \text{and} \quad \boldsymbol{b}_2 = \begin{bmatrix} -2\sqrt{10} & -3\sqrt{10} \end{bmatrix}^T, \\
&\text{for} \quad \boldsymbol{e}_c = \begin{bmatrix} -\frac{3\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \end{bmatrix}^T: \\
&\qquad \boldsymbol{b}_1 = \begin{bmatrix} \frac{\sqrt{10}}{3} & -2\sqrt{10} \end{bmatrix}^T \quad \text{and} \quad \boldsymbol{b}_2 = \begin{bmatrix} 2\sqrt{10} & 3\sqrt{10} \end{bmatrix}^T.
\end{aligned} \tag{40}$$

In above solutions, both pairs of vectors $\vec{\boldsymbol{e}}_c$ as well as the pair of vectors $\vec{\boldsymbol{b}}_1$ and $\vec{\boldsymbol{b}}_2$ represent opposite vectors. For each solution condition (17) is satisfy, namely

$$\vec{\boldsymbol{b}}_1 \vec{\boldsymbol{e}}_c = \vec{\boldsymbol{b}}_2 \vec{\boldsymbol{e}}_c = \alpha = -3. \tag{41}$$

For the first solution, through the points $B_1[b_{11}\ b_{12}]$ and $B_2[b_{21}\ b_{22}]$ one can plot a straight line. Equation of straight line $x_1 - \frac{1}{3}x_2 + \sqrt{10} = 0$, calculated from the series expansions of determinant (25)

$$\begin{bmatrix} 1 & x_1 & x_2 \\ 1 & b_{11} & b_{12} \\ 1 & b_{21} & b_{22} \end{bmatrix} = 0, \quad \begin{bmatrix} 1 & x_1 & x_2 \\ 1 & -\frac{\sqrt{10}}{3} & 2\sqrt{10} \\ 1 & -2\sqrt{10} & -3\sqrt{10} \end{bmatrix} = 0, \tag{42}$$

was obtained with a slope $m = 3$.

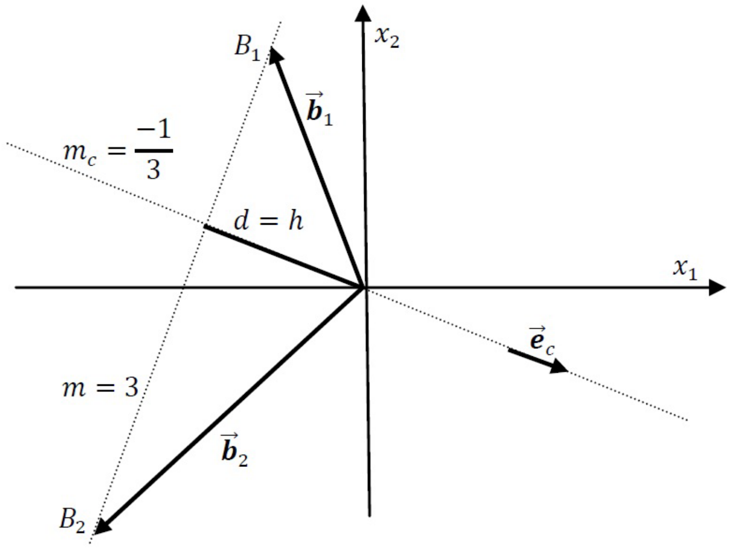

Fig. 1: Geometric interpretation of vectors of a matrix in $\mathbb{R}^2$.

Slope in the direction of the versor $\vec{e}_c$ is $m_c = -\frac{1}{3}$. The condition $m\, m_c = -1$ is satisfied, meaning that eigenversor of matrix $\boldsymbol{A}$ is normal to straight line through the points B_1 and B_2. From Fig. 1 one can conclude that

(43) $$|\alpha| = |\vec{\boldsymbol{b}}_1 \vec{\boldsymbol{e}}_c| = |\vec{\boldsymbol{b}}_2 \vec{\boldsymbol{e}}_c| = d = h,$$

so absolute eigenvalue of matrix $\boldsymbol{A}$ is equal to altitude h of triangle built on vectors $\vec{\boldsymbol{b}}_1$ and $\vec{\boldsymbol{b}}_2$, having triangle base lied on a line B_1B_2 and equal to distance d from the origin to the straight line B_1B_2.

3.2. Eigenvalue $\alpha = 5$

From equation (22) after substituting for the matrix (23) for $\alpha = 5$ one can obtain

(44) $$\begin{bmatrix} -6 & 6 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0.$$

Hence and from the condition (11) is obtained system of equations:

(45) $$\begin{cases} c_1 - c_2 = 0 \\ c_1^2 + c_2^2 = 1 \end{cases} .$$

Two pairs of numbers, which are coordinates of eigenvectors $\boldsymbol{e}_c = [c_1\ c_2]^T$ of matrix $\boldsymbol{A}$, are solution of that system of equations in the form

(46) $$\boldsymbol{e}_c = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T \quad \text{and} \quad \boldsymbol{e}_c = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}^T .$$

Comparing values of the matrix's elements (6) and (13) one obtains coordinates of vectors $\vec{\boldsymbol{b}}_1$ and $\vec{\boldsymbol{b}}_2$ in the matrix form of their coordinates for both eigenvectors

$$\begin{aligned}
&\text{for} \quad \boldsymbol{e}_c = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T : \\
&\qquad \boldsymbol{b}_1 = \begin{bmatrix} -\sqrt{2} & 6\sqrt{2} \end{bmatrix}^T \quad \text{and} \quad \boldsymbol{b}_2 = \begin{bmatrix} 2\sqrt{2} & 3\sqrt{2} \end{bmatrix}^T, \\
&\text{for} \quad \boldsymbol{e}_c = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}^T : \\
&\qquad \boldsymbol{b}_1 = \begin{bmatrix} \sqrt{2} & -6\sqrt{2} \end{bmatrix}^T \quad \text{and} \quad \boldsymbol{b}_2 = \begin{bmatrix} -2\sqrt{2} & -3\sqrt{2} \end{bmatrix}^T.
\end{aligned} \tag{47}$$

In above solutions, both pairs of vectors $\vec{\boldsymbol{e}}_c$ as well as the pair of vectors $\vec{\boldsymbol{b}}_1$ and $\vec{\boldsymbol{b}}_2$ represent opposite vectors. For each solution condition (17) is satisfy, namely

$$\vec{\boldsymbol{b}}_1 \vec{\boldsymbol{e}}_c = \vec{\boldsymbol{b}}_2 \vec{\boldsymbol{e}}_c = \alpha = 5. \tag{48}$$

For the first solution, through the points $B_1[b_{11}\ b_{12}]$ and $B_2[b_{21}\ b_{22}]$ one can plot a straight line. Equation of straight line $\sqrt{2}x_1 + \sqrt{2}x_2 - 2 = 0$, calculated from the series expansions of determinant (25)

$$\begin{bmatrix} 1 & x_1 & x_2 \\ 1 & b_{11} & b_{12} \\ 1 & b_{21} & b_{22} \end{bmatrix} = 0, \quad \begin{bmatrix} 1 & x_1 & x_2 \\ 1 & -\sqrt{2} & 6\sqrt{2} \\ 1 & 2\sqrt{2} & 3\sqrt{2} \end{bmatrix} = 0, \tag{49}$$

was obtained with a slope $m = -1$.

Slope in the direction of the versor $\vec{\boldsymbol{e}}_c$ is $m_c = 1$. The condition $m\ m_c = -1$ is satisfied, meaning that eigenversor of matrix $\boldsymbol{A}$ is normal to straight line through the points B_1 and B_2.

4. The example in $\mathbb{R}^3$

Assumed a square matrix with the following elements

$$\boldsymbol{A} = \begin{bmatrix} -3 & -4 & -4 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}, \quad \det A = 6. \tag{50}$$

We use the form of equation (22). The existence condition of solutions of equation (22) has the form $\det \boldsymbol{A}_\alpha = 0$ and its series expansions gives the characteristic equation

$$\det \boldsymbol{A}_\alpha = \det \begin{bmatrix} -3-\alpha & -4 & -4 \\ 1 & 1-\alpha & 2 \\ 0 & 2 & 2-\alpha \end{bmatrix} = 0 \quad \longmapsto \quad -\alpha^3 + 7\alpha + 6 = 0. \tag{51}$$

This yields to a conclusion that matrix has three eigenvalues $\alpha = 3$, $\alpha = -1$ and $\alpha = -2$.

4.1. Eigenvalue $\alpha = 3$

From equation (22) after substituting for the matrix (23) for $\alpha = 3$ one can obtain

$$\begin{bmatrix} -6 & -4 & -4 \\ 1 & -2 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0. \tag{52}$$

Hence and from the condition (11) is obtained system of equations

$$\begin{cases} 3c_1 + 2c_2 + 2c_3 = 0 \\ c_1 - 2c_2 + 2c_3 = 0 \\ 2c_2 - c_3 = 0 \\ c_1^2 + c_2^2 + c_3^2 = 1 \end{cases}. \tag{53}$$

Two triples of numbers, which are coordinates of eigenvectors $\boldsymbol{e}_c = [c_1\ c_2\ c_3]^T$ of matrix $\boldsymbol{A}$, are solution of that system of equations in the form

$$\boldsymbol{e}_c = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}^T \quad \text{and} \quad \boldsymbol{e}_c = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}^T. \tag{54}$$

Comparing values of the matrix's elements (6) and (13) one obtains coordinates of vectors $\vec{\boldsymbol{b}}_1$, $\vec{\boldsymbol{b}}_2$ and $\vec{\boldsymbol{b}}_3$ in the matrix form of their coordinates for both eigenvectors

$$\begin{aligned} &\text{for} \quad \boldsymbol{e}_c = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}^T: && \boldsymbol{b}_1 = \begin{bmatrix} \frac{9}{2} & 6 & 6 \end{bmatrix}^T, \\ &\qquad \boldsymbol{b}_2 = \begin{bmatrix} 3 & 3 & 6 \end{bmatrix}^T, && \boldsymbol{b}_3 = \begin{bmatrix} 0 & 3 & 3 \end{bmatrix}^T, \\ &\text{for} \quad \boldsymbol{e}_c = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}^T: && \boldsymbol{b}_1 = \begin{bmatrix} -\frac{9}{2} & -6 & -6 \end{bmatrix}^T, \\ &\qquad \boldsymbol{b}_2 = \begin{bmatrix} -3 & -3 & -6 \end{bmatrix}^T, && \boldsymbol{b}_3 = \begin{bmatrix} 0 & -3 & -3 \end{bmatrix}^T. \end{aligned} \tag{55}$$

Both pairs of vectors $\vec{\boldsymbol{e}}_c$ as well as the pair of vectors $\vec{\boldsymbol{b}}_1$, $\vec{\boldsymbol{b}}_2$ and $\vec{\boldsymbol{b}}_3$ occurring in above solutions, satisfy condition (17), namely

$$\vec{\boldsymbol{b}}_1 \vec{\boldsymbol{e}}_c = \vec{\boldsymbol{b}}_2 \vec{\boldsymbol{e}}_c = \vec{\boldsymbol{b}}_3 \vec{\boldsymbol{e}}_c = \alpha = 3. \tag{56}$$

For the first solution, through the points $B_1\ [b_{11}\ \ b_{12}\ \ b_{13}]$, $B_2\ [b_{21}\ \ b_{22}\ \ b_{23}]$, $B_3\ [b_{31}\ \ b_{32}\ \ b_{33}]$ one can plot a plane. Equation of plane calculated from the series expansions of determinant (25)

$$\begin{bmatrix} 1 & x_1 & x_2 & x_3 \\ 1 & b_{11} & b_{12} & b_{13} \\ 1 & b_{21} & b_{22} & b_{23} \\ 1 & b_{31} & b_{32} & b_{33} \end{bmatrix} = 0, \quad \begin{bmatrix} 1 & x_1 & x_2 & x_3 \\ 1 & \frac{9}{2} & 6 & 6 \\ 1 & 3 & 3 & 6 \\ 1 & 0 & 3 & 3 \end{bmatrix} = 0, \tag{57}$$

has a form

$$-2x_1 + x_2 + 2x_3 - 18 = 0. \tag{58}$$

The versor, which is normal to this plane, has the coordinates $\boldsymbol{e}_\pi = [-\frac{2}{3}\ \frac{1}{3}\ \frac{2}{3}]^T$ and is the same as eigenversor $\boldsymbol{e}_c = [-\frac{2}{3}\ \frac{1}{3}\ \frac{2}{3}]^T$ of matrix $\boldsymbol{A}$ corresponding to this plane. This means that the eigenversor $\vec{\boldsymbol{e}}_c$ is orthogonal to the plane π. The considered case axis of an umbrella is the axis corresponding to the eigenversor of matrix $\boldsymbol{A}$ and the plane of the umbrella forms a plane π, whose position in space specifies the same versor $\boldsymbol{e}_c = [-\frac{2}{3}\ \ \frac{1}{3}\ \ \frac{2}{3}]^T$. Vector $\vec{\boldsymbol{b}}$ (31) of matrix $\boldsymbol{A}$ has the coordinates

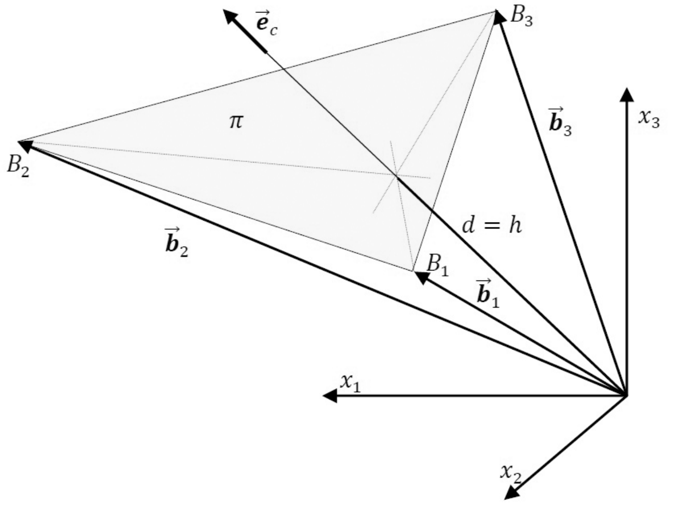

Fig. 2: Geometric interpretation of vectors of a matrix in $\mathbb{R}^3$.

$\boldsymbol{b} = [\frac{15}{2}\ \ 12\ \ 15]^T$. Vector projection $\vec{\boldsymbol{b}}_c$ onto the direction of eigenvector $\vec{\boldsymbol{e}}_c$ is a vector $\vec{\boldsymbol{b}}_c = n\alpha\ \vec{\boldsymbol{e}}_c$ with the coordinates $\boldsymbol{b}_c = [-6\ \ 3\ \ 6]^T$, and vector projection $\vec{\boldsymbol{b}}_\pi$ onto the plane π, calculate from the matrix (33) or (34), has the coordinates $\boldsymbol{b}_\pi = [\frac{27}{2}\ \ 9\ \ 9]^T$. Fig. 2 shows the set of vectors of the matrix $\boldsymbol{A}$: eigenversor $\vec{\boldsymbol{e}}_c$, vectors $\vec{\boldsymbol{b}}_1$, $\vec{\boldsymbol{b}}_2$ and $\vec{\boldsymbol{b}}_3$ defining the umbrella plane π normal to the vector $\vec{\boldsymbol{e}}_c$ and the distance $d = h$ such that

(59) $$d = h = |\vec{\boldsymbol{b}}_1 \vec{\boldsymbol{e}}_c| = |\vec{\boldsymbol{b}}_2 \vec{\boldsymbol{e}}_c| = |\vec{\boldsymbol{b}}_3 \vec{\boldsymbol{e}}_c| = |\alpha| = 3,$$

Thus its length is equal to the absolute eigenvalue of matrix $\boldsymbol{A}$.

4.2. Eigenvalue $\alpha = -1$ and $\alpha = -2$

From equation (22) after substituting for the matrix (23) for $\alpha = -1$ one can obtain

(60) $$\begin{bmatrix} -2 & -4 & -4 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0.$$

Hence and from the condition (11) is obtained system of equations

(61) $$\begin{cases} c_1 + 2c_2 + 2c_3 = 0 \\ \qquad\; 2c_2 + 3c_3 = 0 \\ c_1^2 + c_2^2 + c_3^2 = 1 \end{cases}.$$

Two triples of numbers, which are coordinates of eigenvectors $\boldsymbol{e}_c = [c_1\ c_2\ c_3]^T$ of matrix $\boldsymbol{A}$, are solution of that system of equations in the form

$$(62)\quad \boldsymbol{e}_c = \begin{bmatrix} -\frac{2\sqrt{17}}{17} & \frac{3\sqrt{17}}{17} & -\frac{2\sqrt{17}}{17} \end{bmatrix}^T, \quad \boldsymbol{e}_c = \begin{bmatrix} \frac{2\sqrt{17}}{17} & -\frac{3\sqrt{17}}{17} & \frac{2\sqrt{17}}{17} \end{bmatrix}^T.$$

Coordinates of vectors $\vec{\boldsymbol{b}}_1$, $\vec{\boldsymbol{b}}_2$ and $\vec{\boldsymbol{b}}_3$ in the matrix form of their coordinates for both eigenvectors:

$$(63)\quad \begin{aligned} &\boldsymbol{e}_c = \begin{bmatrix} \frac{-2\sqrt{17}}{17} & \frac{3\sqrt{17}}{17} & \frac{-2\sqrt{17}}{17} \end{bmatrix}^T: && \boldsymbol{b}_1 = \begin{bmatrix} \frac{3\sqrt{17}}{17} & 2\sqrt{17} & 2\sqrt{17} \end{bmatrix}^T, \\ &\boldsymbol{b}_2 = \begin{bmatrix} \frac{\sqrt{17}}{3} & \frac{\sqrt{17}}{3} & \frac{2\sqrt{17}}{3} \end{bmatrix}^T, && \boldsymbol{b}_3 = \begin{bmatrix} 0 & -\sqrt{17} & -\sqrt{17} \end{bmatrix}^T, \\ &\boldsymbol{e}_c = \begin{bmatrix} \frac{2\sqrt{17}}{17} & \frac{-3\sqrt{17}}{17} & \frac{2\sqrt{17}}{17} \end{bmatrix}^T: && \boldsymbol{b}_1 = \begin{bmatrix} \frac{-3\sqrt{17}}{17} & -2\sqrt{17} & -2\sqrt{17} \end{bmatrix}^T, \\ &\boldsymbol{b}_2 = \begin{bmatrix} \frac{-\sqrt{17}}{3} & \frac{-\sqrt{17}}{3} & \frac{-2\sqrt{17}}{3} \end{bmatrix}^T, && \boldsymbol{b}_3 = \begin{bmatrix} 0 & \sqrt{17} & \sqrt{17} \end{bmatrix}^T, \end{aligned}$$

As an elegant application we can obtain the relation shown in Fig. 3 [6].

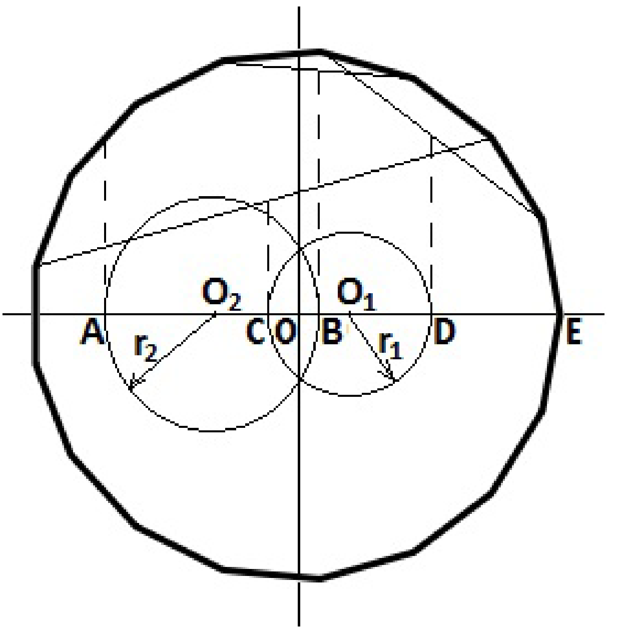

Fig. 3: The length the joint chord of the circles with diameters AB and CD equals $\frac{1}{2}|\mathrm{OE}|$.

Both pairs of vectors $\vec{\boldsymbol{e}}_c$ as well as the pair of vectors $\vec{\boldsymbol{b}}_1$, $\vec{\boldsymbol{b}}_2$ and $\vec{\boldsymbol{b}}_3$ occurring in above solutions, satisfy condition (17), namely

$$\vec{\boldsymbol{b}}_1 \vec{\boldsymbol{e}}_c = \vec{\boldsymbol{b}}_2 \vec{\boldsymbol{e}}_c = \vec{\boldsymbol{b}}_3 \vec{\boldsymbol{e}}_c = \alpha = -1. \tag{64}$$

From equation (22) after substituting for the matrix (23) for $\alpha = -1$ one can obtain

$$\begin{bmatrix} -1 & -4 & -4 \\ 1 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0. \tag{65}$$

Hence and from the condition (11) is obtained system of equations

$$\begin{cases} c_1 & + & 4c_2 & + & 4c_3 & = 0 \\ c_1 & + & 3c_2 & + & 2c_3 & = 0 \\ & & c_2 & + & 2c_3 & = 0 \\ c_1^2 & + & c_2^2 & + & c_3^2 & = 1 \end{cases}, \tag{66}$$

Two triples of numbers, which are coordinates of eigenvectors $\boldsymbol{e}_c = [c_1 \ c_2 \ c_3]^T$ of matrix $\boldsymbol{A}$, are solution of that system of equations in the form

$$\boldsymbol{e}_c = \begin{bmatrix} \frac{-4\sqrt{21}}{21} & \frac{2\sqrt{21}}{21} & \frac{-\sqrt{21}}{21} \end{bmatrix}^T \quad \text{and} \quad \boldsymbol{e}_c = \begin{bmatrix} \frac{4\sqrt{21}}{21} & \frac{-2\sqrt{21}}{21} & \frac{\sqrt{21}}{21} \end{bmatrix}^T. \tag{67}$$

Coordinates of vectors $\vec{\boldsymbol{b}}_1$, $\vec{\boldsymbol{b}}_2$ and $\vec{\boldsymbol{b}}_3$ in the matrix form of their coordinates for both eigenvectors has a form

$$\begin{aligned} &\boldsymbol{e}_c = \begin{bmatrix} \frac{-4\sqrt{21}}{21} & \frac{2\sqrt{21}}{21} & \frac{-\sqrt{21}}{21} \end{bmatrix}^T : \quad \boldsymbol{b}_1 = \begin{bmatrix} \frac{3\sqrt{21}}{4} & \sqrt{21} & \sqrt{21} \end{bmatrix}^T, \\ &\boldsymbol{b}_2 = \begin{bmatrix} \frac{\sqrt{21}}{2} & \frac{\sqrt{21}}{2} & \sqrt{21} \end{bmatrix}^T, \qquad \boldsymbol{b}_3 = \begin{bmatrix} 0 & -2\sqrt{21} & -2\sqrt{21} \end{bmatrix}^T, \\ &\boldsymbol{e}_c = \begin{bmatrix} \frac{4\sqrt{21}}{21} & \frac{-2\sqrt{21}}{21} & \frac{\sqrt{21}}{21} \end{bmatrix}^T : \quad \boldsymbol{b}_1 = \begin{bmatrix} \frac{-3\sqrt{21}}{4} & -\sqrt{21} & -\sqrt{21} \end{bmatrix}^T, \\ &\boldsymbol{b}_2 = \begin{bmatrix} \frac{-\sqrt{21}}{2} & \frac{-\sqrt{21}}{2} & -\sqrt{21} \end{bmatrix}^T, \quad \boldsymbol{b}_3 = \begin{bmatrix} 0 & 2\sqrt{21} & 2\sqrt{21} \end{bmatrix}^T, \end{aligned} \tag{68}$$

Both pairs of vectors $\vec{\boldsymbol{e}}_c$ as well as the pair of vectors $\vec{\boldsymbol{b}}_1$, $\vec{\boldsymbol{b}}_2$ and $\vec{\boldsymbol{b}}_3$ occurring in above solutions, satisfy condition (17), namely

$$\vec{\boldsymbol{b}}_1 \vec{\boldsymbol{e}}_c = \vec{\boldsymbol{b}}_2 \vec{\boldsymbol{e}}_c = \vec{\boldsymbol{b}}_3 \vec{\boldsymbol{e}}_c = \alpha = -2. \tag{69}$$

In both cases, the planes through the points B_1 $[b_{11} \ \ b_{12} \ \ b_{13}]$, B_2 $[b_{21} \ \ b_{22} \ \ b_{23}]$, B_3 $[b_{31} \ \ b_{32} \ \ b_{33}]$ are normal to the eigenversors. Modules of eigenvalues of matrix satisfy the condition

$$|\alpha| = |\vec{\boldsymbol{b}}_1 \vec{\boldsymbol{e}}_c| = |\vec{\boldsymbol{b}}_2 \vec{\boldsymbol{e}}_c| = |\vec{\boldsymbol{b}}_3 \vec{\boldsymbol{e}}_c| = d = h, \tag{70}$$

where d is plane distance from the origin and h is height of tetrahedron with edges formed by the vectors $\vec{\boldsymbol{b}}_1$, $\vec{\boldsymbol{b}}_2$ and $\vec{\boldsymbol{b}}_3$.

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WEKTORY MACIERZY KWADRATOWYCH W PRZESTRZENI $\mathbb{R}^n$

S t r e s z c z e n i e

Przedstawiono różne aspekty technik obliczeniowych dla iloczynu zewnętrznego dwu wektorów w związku z zagadnieniem wartości własnych.

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In memory of
Professor Promarz M. Tamrazov

Oleg F. Gerus

AN ESTIMATE FOR THE MODULUS OF CONTINUITY OF A QUATERNION SINGULAR CAUCHY INTEGRAL ON A REGULAR SURFACE

Summary

Sufficient conditions for existence of a quaternion singular Cauchy integral in the theory of α-hyperholomorphic functions, defined on a closed rectifiable regular surface in the space $\mathbb{R}^3$, is established and an upper estimate for its modulus of continuity is proved in terms of the modulus of continuity of the integrand.

Keywords and phrases: quaternion Cauchy integral, modulus of convexity

1. Introduction

For the first time A. Zygmund [1] proved an estimate for the modulus of continuity of a trigonometrically conjugate function on the real axis. It is equivalent to the estimate for the modulus of continuity of a singular Cauchy integral on a circle. This estimate yields, in particular, the Plemelj-Privalov theorem on the invariance of Hölder classes with respect to a singular Cauchy integral. The A. Zygmund estimate was extended to broader classes of curves in works of L. G. Magnaradze [2,3], A. A. Babaev and V. V. Salaev [4–6], P. M. Tamrazov [7,8], O. F. Gerus [9–11], T. S. Salimov [12], E. M. Dyn'kin [13]. In particular, it was established that the broadest class of curves (see [6,9]) for the estimate to have the same form as on a circle is the class of regular curves (for which the measure of the part of a curve that enters the disk does not exceed a constant multiplied by the radius of the disk). For more general curves (see [6, 9–13]) the majorant worsens and depends on the curve.

In [14] we considered a generalization of the Cauchy type integral in the theory of so-called α-hyperholomorphic functions acting from the space $\mathbb{R}^2$ equipped with a certain quaternion structure to the algebra of complex quaternions. We proved formulas for the boundary values of the integral on closed piecewise Lyapunov curves and the Plemelj-Privalov theorem for appropriate singular integral. In the work [15] we proved formulas for the boundary values of the quaternion Cauchy type integral on closed Jordan rectifiable curves and in the work [16] we obtained an estimate for the modulus of continuity of the appropriate singular integral.

In [17] an analog of the A. Zygmund estimate was proved for a singular Cauchy integral in the theory of quaternion hyperholomorphic functions on surfaces of the space $\mathbb{R}^3$, equipped with a certain quaternion structure. The aim of this work is to obtain an estimate for the modulus of continuity of a singular Cauchy integral in the theory of α-hyperholomorphic functions (see [18]) on regular surfaces in the space $\mathbb{R}^3$.

2. Quaternions. Quaternion differentiable functions in $\mathbb{R}^3$

Let $\mathbb{H} = \mathbb{H}(\mathbb{R})$ and $\mathbb{H}(\mathbb{C})$ denote, respectively, the algebras of real and complex quaternions

$$a = \sum_{k=0}^{3} a_k \boldsymbol{i_k}, \quad \text{where} \quad \{a_k\}_{k=0}^{3} \subset \mathbb{R}$$

for real quaternions, $\{a_k\}_{k=0}^{3} \subset \mathbb{C}$ for complex quaternions, $\boldsymbol{i}_0 = 1$, $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, and $\boldsymbol{i}_3$ are imaginary quaternion units with the multiplication rule

$${\boldsymbol{i}_1}^2 = {\boldsymbol{i}_2}^2 = {\boldsymbol{i}_3}^2 = \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = -1.$$

Under the modulus of a complex quaternion we understand its Euclidean norm

$$|a| := \sqrt{\sum_{k=0}^{3} |a_k|^2}.$$

For complex quaternions hold relations $|a|^2 \neq a\overline{a}$ and

$$|ab| \leqslant \sqrt{2}\,|a|\,|b| \tag{1}$$

(see Lemma 2.1 of the paper [15]).

Let $z := z_1\boldsymbol{i}_1 + z_2\boldsymbol{i}_2 + z_3\boldsymbol{i}_3$ be a real quaternion, let Ω be a domain in the space $\mathbb{R}^3$, and let $\alpha \in \mathbb{C}$. For functions $f : \Omega \to \mathbb{H}(\mathbb{C})$ having the partial derivatives of the second order consider the Helmholtz operator

$$\Delta_{\alpha^2} := \sum_{k=0}^{2} \frac{\partial^2}{\partial z_k^2} + \alpha^2 I,$$

where I is the identical operator. The following factorizations hold (see [18]):

$$\Delta_{\alpha^2} = -D_\alpha\, D_{-\alpha} = -{}_\alpha D\,{}_{-\alpha} D,$$

where

$$D_\alpha[f] := \sum_{k=1}^{3} \boldsymbol{i_k} \frac{\partial f}{\partial z_k} + \alpha f,$$

$$_\alpha D[f] := \sum_{k=1}^{3} \frac{\partial f}{\partial z_k} \boldsymbol{i_k} + \alpha f.$$

The fundamental solution of the Helmholtz operator in the space $\mathbb{R}^3$ has the form (see [19]):

$$\mathcal{E}_\alpha(z) = -\frac{e^{-i\alpha|z|}}{4\pi|z|}.$$

The function

$$K_\alpha(z) := -D_{-\alpha}[\mathcal{E}_\alpha](z) = \left(\alpha + \frac{z}{|z|^2} + i\alpha\frac{z}{|z|}\right)\mathcal{E}_\alpha(z),$$

which is a fundamental solution of operators D_α and $_\alpha D$, is called the quaternion Cauchy kernel just as the Cauchy kernel $\dfrac{1}{2\pi z}$ in complex analysis is a fundamental solution of the Cauchy-Riemann operator

$$\overline{\partial} := \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

3. Quaternion Cauchy type integral and singular Cauchy integral

Let Γ be a rectifiable surface in the space $\mathbb{R}^3$, i.e., Γ is the image of a bounded set from $\mathbb{R}^2$ under its Lipschitz mapping into $\mathbb{R}^3$. The particular case of a rectifiable surface is a piecewise surface. A surface Γ is called a closed surface if it divides the space on two domains: interior Ω^+ with respect to Γ and exterior Ω^-. It is known (see [20]) that the tangent plane exists in almost every point of a rectifiable surface. Denote by $\nu(\zeta) := \nu_1(\zeta)\boldsymbol{i}_1 + \nu_2(\zeta)\boldsymbol{i}_2 + \nu_3(\zeta)\boldsymbol{i}_3$ the unit normal vector to oriented surface Γ in that points $\zeta \in \Gamma$, where it exists. We choose the orientation of a closed surface for the normal vector to be directed into the domain Ω^-.

For a closed Jordan rectifiable surface $\Gamma \subset \mathbb{R}^3$ and for a continuous function $f : \Gamma \to \mathbb{H}(\mathbb{C})$ the quaternion Cauchy type integral is defined by the formula (see [18])

$$\Phi_\alpha[f](z) := \int_\Gamma K_\alpha(\zeta - z)\,\nu(\zeta)\,f(\zeta)\,ds_\zeta, \qquad z \in \mathbb{R}^3 \setminus \Gamma,$$

where ds_ζ is the surface square element.

Let $\Gamma_{z,\delta} := \{\zeta{\in}\Gamma : |\zeta - z| \leqslant \delta\}$. The object of research in this work is the singular integral

$$F_\alpha[f](t) := \lim_{\delta\to 0} \int\limits_{\Gamma\setminus\Gamma_{t,\delta}} K_\alpha(\zeta - t)\,\nu(\zeta)\,(f(\zeta) - f(t))\,ds_\zeta, \qquad t \in \Gamma,$$

using for representation of the boundary values of the Cauchy type integral Φ_α (in the case of $\alpha = 0$ see [21]).

Let us denote

$$K_{\alpha,1}(z) = \alpha\,\mathcal{E}_\alpha(z),$$

$$K_{\alpha,2}(z) = \frac{z}{|z|^2}\,\mathcal{E}_\alpha(z),$$

$$K_{\alpha,3}(z) = i\alpha\frac{z}{|z|}\,\mathcal{E}_\alpha(z).$$

Then

$$K_\alpha = K_{\alpha,1} + K_{\alpha,2} + K_{\alpha,3}$$

and so

$$F_\alpha[f](t) = \sum_{k=1}^{3} F_{\alpha,k}[f](t), \tag{2}$$

where

$$F_{\alpha,k}[f](t) := \lim_{\delta\to 0} \int\limits_{\Gamma\setminus\Gamma_{t,\delta}} K_{\alpha,k}(\zeta - t)\,\nu(\zeta)\,(f(\zeta) - f(t))\,ds_\zeta.$$

Assume that $\delta > 0$, $E \subset \mathbb{R}^3$,

$$\omega_\Gamma(f,\delta) := \sup_{\substack{|z_1 - z_2| \leqslant \delta \\ \{z_1; z_2\} \subset \Gamma}} |f(z_1) - f(z_2)|$$

is the modulus of continuity of a function f on Γ, $\theta_z(\delta) := \operatorname{mes}\Gamma_{z,\delta}$ is the surface measure of the set $\Gamma_{z,\delta}$.

Later on the symbol c in estimates will denote positive absolute constants (possibly different).

Definition. A closed Jordan rectifiable surface Γ is called a regular surface, if there exists a positive constant K, such as for all $z \in \Gamma$ and for all $\delta > 0$ the next inequality holds true:

$$\theta_z(\delta) \leqslant K\delta^2. \tag{3}$$

Theorem 1. *Let Γ be a regular surface and a function $f : \Gamma \to \mathbb{H}(\mathbb{C})$ satisfies the condition*

$$\int\limits_0^d \frac{\omega_\Gamma(f,x)}{x}\,dx < +\infty, \tag{4}$$

where d is the diameter of the surface Γ. Then the integral $F_\alpha[f]$ exists in every point of the space Γ and the estimate

$$\omega_\Gamma(F_\alpha,\delta)\leqslant cKe^{3|\alpha|d}\left((1+2d|\alpha|)\int\limits_0^{2d}\frac{\omega_\Gamma(f,x)}{x\left(1+\dfrac{x}{\delta}\right)}dx+|\alpha|^2\delta\int\limits_\delta^{2d}\omega_\Gamma(f,x)\,dx\right)$$

holds true.

Proof. It follows from (4) that $\omega_\Gamma(f,\delta)=o(1)$ when $\delta\to 0$. Thus the function f is uniformly continuous on the surface Γ.

In order to prove the existence of the integral F_α we may prove only the existence of the integral $F_{\alpha,2}$ as long as $K_{\alpha,2}$ is the main term of the Cauchy kernel K_α.

Let us fix any point $t\in\Gamma$. Let $n\in\mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers), $d_n=\dfrac{d}{2^{n-1}}$, $\Gamma_n:=\Gamma_{t,d_n}\setminus\Gamma_{t,d_{n+1}}$. Then

$$F_{\alpha,2}=\sum_{n=1}^{\infty}I_2(\Gamma_n),$$

where

$$I_2(\Gamma_n):=\int\limits_{\Gamma_n}K_{\alpha,2}(\zeta-t)\nu(\zeta)(f(\zeta)-f(t))ds_\zeta.$$

Taking into account the inequality (1), we have

$$|I_2(\Gamma_n)|\leqslant\frac{1}{2\pi}\int\limits_{\Gamma_n}e^{|\alpha|\,|\zeta-t|}\frac{|f(\zeta)-f(t)|}{|\zeta-t|^2}\,ds_\zeta\leqslant$$

$$\leqslant\frac{1}{2\pi}e^{|\alpha|\,d_n}\frac{\omega_\Gamma(f,d_n)}{d_{n+1}^2}\left(\theta_t(d_n)-\theta_t(d_{n+1})\right)\leqslant\frac{2}{\pi}e^{|\alpha|\,d_n}\int\limits_{d_{n+1}}^{d_n}\frac{\omega_\Gamma(f,2y)}{y^2}\,d\theta_t(y).$$

Therefore, owing to the monotony of the modulus of continuity $\omega_{\Gamma,f}$ and the inequality (3), we obtain

$$\sum_{n=1}^{\infty}|I(\Gamma_n)|\leqslant\frac{2}{\pi}e^{|\alpha|\,d}\int\limits_0^{d}\frac{\omega_\Gamma(f,2y)}{y^2}\,d\theta_t(y)\leqslant\frac{16}{3\pi}Ke^{|\alpha|\,d}\int\limits_0^{4d}\frac{\omega_\Gamma(f,x)}{x}\,dx. \tag{5}$$

So the condition (4) implies the absolute convergence of the series $\sum\limits_{n=1}^{\infty}I(\Gamma_n)$ and consequently the convergence of the singular integral F_α.

Let us estimate the modulus of continuity of the integral $F_{\alpha,2}$. Let $0<\delta\leqslant\frac{d}{3}$. Consider any points $\{t^{(1)};t^{(2)}\}\subset\Gamma$ such as $\delta_1:=|t^{(1)}-t^{(2)}|\leqslant\delta$. Let us define $\gamma:=\Gamma_{t^{(1)},2\delta_1}$. Then

(6)

$$4\pi\left(F_{\alpha,2}\left(t^{(2)}\right)-F_{\alpha,2}\left(t^{(1)}\right)\right)=\int\limits_{\gamma}e^{-i\alpha|\zeta-t^{(1)}|}\frac{\zeta-t^{(1)}}{|\zeta-t^{(1)}|^3}\nu(\zeta)(f(\zeta)-f(t^{(1)}))ds_\zeta-$$
$$-\int\limits_{\gamma}e^{-i\alpha|\zeta-t^{(2)}|}\frac{\zeta-t^{(2)}}{|\zeta-t^{(2)}|^3}\nu(\zeta)(f(\zeta)-f(t^{(2)}))ds_\zeta+$$
$$+\int\limits_{\Gamma\setminus\gamma}e^{-i\alpha|\zeta-t^{(1)}|}\frac{\zeta-t^{(1)}}{|\zeta-t^{(1)}|^3}\nu(\zeta)(f(t^{(2)})-f(t^{(1)}))ds_\zeta+$$
$$+\int\limits_{\Gamma\setminus\gamma}\left(e^{-i\alpha|\zeta-t^{(1)}|}\frac{\zeta-t^{(1)}}{|\zeta-t^{(1)}|^3}-e^{-i\alpha|\zeta-t^{(2)}|}\frac{\zeta-t^{(2)}}{|\zeta-t^{(2)}|^3}\right)\nu(\zeta)(f(\zeta)-f(t^{(2)}))ds_\zeta=:$$
$$=:I_1+I_2+I_3+I_4.$$

Similarly to the estimate (5) we obtain

$$|I_1|\leqslant 8e^{2|\alpha|\delta_1}\int\limits_0^{2\delta_1}\frac{\omega_\Gamma(f,2y)}{y^2}\,d\theta_{t^{(1)}}(y)\leqslant\frac{64}{3}Ke^{2|\alpha|\delta_1}\int\limits_0^{8\delta_1}\frac{\omega_\Gamma(f,x)}{x}\,dx, \tag{7}$$

$$|I_2|\leqslant 8e^{3|\alpha|\delta_1}\int\limits_0^{3\delta_1}\frac{\omega_\Gamma(f,2y)}{y^2}\,d\theta_{t^{(2)}}(y)\leqslant\frac{64}{3}Ke^{3|\alpha|\delta_1}\int\limits_0^{12\delta_1}\frac{\omega_\Gamma(f,x)}{x}\,dx. \tag{8}$$

In order to estimate $|I_3|$ we complement the surface $\Gamma\setminus\gamma$ to a closed surface by the surface S_γ, which consists of parts of the sphere having the radius $2\delta_1$, the center $t^{(1)}$, the common edge with $\Gamma\setminus\gamma$, and having the normal vector oriented to the center of the sphere.

Owing to the Gauss-Ostrogradsky formula, the integral along closed surface $\Gamma\setminus\gamma+S_\gamma$ is equal zero. Therefore

$$I_3=-\int\limits_{S_\gamma}e^{-i\alpha|\zeta-t^{(1)}|}\frac{\zeta-t^{(1)}}{|\zeta-t^{(1)}|^3}\nu(\zeta)(f(t^{(2)})-f(t^{(1)}))ds_\zeta=$$
$$=\frac{\mathrm{mes}S_\gamma}{4\delta_1^2}e^{-i\alpha 2\delta_1}(f(t^{(2)})-f(t^{(1)}).$$

Thus

$$|I_3|\leqslant 4\pi e^{2|\alpha|\delta_1}\omega_\Gamma(f,\delta_1)\leqslant 8\pi e^{2|\alpha|\delta_1}\int\limits_0^{2\delta_1}\frac{\omega_\Gamma(f,x)}{x}\,dx. \tag{9}$$

Let us estimate $|I_4|$. Let $\nu\in\mathbb{N}$ such as $d\leqslant 2^\nu\delta_1<2d$, $\widetilde{\Gamma}_n:=(\Gamma_{t^{(2)},\delta_{n+1}}\setminus\Gamma_{t^{(2)},\delta_n})\setminus\gamma$, $\delta_n:=2^{n-1}\delta_1$, $n=1,2,\ldots,\nu$. Then

$$I_4 = \sum_{n=1}^{\nu} \left(I_{4,n,1} + I_{4,n,2} \right), \tag{10}$$

where

$$I_{4,n,1} = \int\limits_{\widetilde{\Gamma}_n} \left(e^{-i\alpha|\zeta - t^{(1)}|} - e^{-i\alpha|\zeta - t^{(2)}|} \right) \frac{\zeta - t^{(2)}}{|\zeta - t^{(2)}|^3} \nu(\zeta)(f(\zeta) - f(t^{(2)})) ds_\zeta, \tag{11}$$

$$I_{4,n,2} = \int\limits_{\widetilde{\Gamma}_n} e^{-i\alpha|\zeta - t^{(1)}|} \left(\frac{\zeta - t^{(1)}}{|\zeta - t^{(1)}|^3} - \frac{\zeta - t^{(2)}}{|\zeta - t^{(2)}|^3} \right) \nu(\zeta)(f(\zeta) - f(t^{(2)})) ds_\zeta. \tag{12}$$

$$\begin{aligned} |I_{4,n,1}| &\leqslant 16\, e^{|\alpha|\, d} |\alpha| \delta_1 \frac{\omega_\Gamma(f, \delta_{n+1})}{(\delta_{n+1})^2} \left(\theta_{t^{(2)}}(\delta_{n+1}) - \theta_{t^{(2)}}(\delta_n) \right) \leqslant \\ &\leqslant 16\, e^{|\alpha|\, d} |\alpha| \delta_1 \int\limits_{\delta_n}^{\delta_{n+1}} \frac{\omega_\Gamma(f, 2y)}{y^2}\, d\theta_{t^{(2)}}(y). \end{aligned} \tag{13}$$

Analogously, by using the inequality (see [22])

$$\left| \frac{a}{|a|^3} - \frac{b}{|b|^3} \right| \leqslant |a - b| \left(\frac{1}{|a|^2 |b|} + \frac{1}{|a| |b|^2} \right)$$

for any quaternions a and b, we obtain

$$\begin{aligned} |I_{4,n,2}| &\leqslant 68\, e^{|\alpha|\, d} \delta_1 \frac{\omega_\Gamma(f, \delta_{n+1})}{(\delta_{n+1})^3} \left(\theta_{t^{(2)}}(\delta_{n+1}) - \theta_{t^{(2)}}(\delta_n) \right) \leqslant \\ &\leqslant 68\, e^{|\alpha|\, d} \delta_1 \int\limits_{\delta_n}^{\delta_{n+1}} \frac{\omega_\Gamma(f, 2y)}{y^3}\, d\theta_{t^{(2)}}(y). \end{aligned} \tag{14}$$

Using relations (6)–(14) we get

$$\omega_\Gamma(F_{\alpha,2}, \delta) \leqslant cK e^{3|\alpha| d} \left(\int\limits_0^{2d} \frac{\omega_\Gamma(f, x)}{x \left(1 + \dfrac{x}{\delta} \right)} dx + |\alpha| \delta \int\limits_\delta^{2d} \frac{\omega_\Gamma(f, x)}{x}\, dx \right). \tag{15}$$

Analogously by using inequalities

$$\left| \frac{1}{|a|} - \frac{1}{|b|} \right| \leqslant \frac{|a - b|}{|a||b|}, \quad \left| \frac{a}{|a|^2} - \frac{b}{|b|^2} \right| \leqslant \frac{|a - b|}{|a||b|}$$

respectively, we obtain the next estimation for continuity moduli of integrals $F_{\alpha,k}$, $k \in \{1; 3\}$:

$$\omega_\Gamma(F_{\alpha,k}, \delta) \leqslant cK e^{3|\alpha| d} \left(|\alpha| \int\limits_0^{2d} \frac{\omega_\Gamma(f, x)}{1 + \dfrac{x}{\delta}} dx + |\alpha|^2 \delta \int\limits_\delta^{2d} \omega_\Gamma(f, x)\, dx \right). \tag{16}$$

It follows from the relations (2), (15), (16) that

$$\omega_\Gamma(F_\alpha,\delta) \leqslant cKe^{3|\alpha|d}\left(\int\limits_0^{2d}\frac{\omega_\Gamma(f,x)}{x\left(1+\dfrac{x}{\delta}\right)}dx+|\alpha|\int\limits_0^{2d}\frac{\omega_\Gamma(f,x)}{1+\dfrac{x}{\delta}}dx+|\alpha|^2\delta\int\limits_\delta^{2d}\omega_\Gamma(f,x)\,dx\right)t$$
$$\leqslant cKe^{3|\alpha|d}\left((1+2d|\alpha|)\int\limits_0^{2d}\frac{\omega_\Gamma(f,x)}{x\left(1+\dfrac{x}{\delta}\right)}dx+|\alpha|^2\delta\int\limits_\delta^{2d}\omega_\Gamma(f,x)\,dx\right).$$

Theorem is proved.

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OSZACOWANIE MODUŁU CIĄGŁOŚCI KWATERNIONOWEJ OSOBLIWEJ CAŁKI CAUCHY'EGO NA POWIERZCHNI REGULARNEJ

S t r e s z c z e n i e

Uzyskujemy wystraczające warunki istnienia kwaternionowej osobliwej całki Cauchy'ego w teorii funkcji α-hiperholomorficznej, określonych na domkniętej prostowalnej powierzchni regularnej w przestrzeni $\mathbb{R}^3$. Ponadto wyznaczamy kres górny odpowiedniego modułu ciągłości w terminach modułu ciągłości funkcji podcałkowej.

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Dedicated to memory
of Professor Hans Grauert

Julian Ławrynowicz, Kiyoharu Nôno, and Osamu Suzuki

BINARY AND TERNARY CLIFFORD ANALYSIS NONCOMMUTATIVE GALOIS EXTENSIONS II

THE CORRESPONDENCE BETWEEN TERNARY AND BINARY FIELD OPERATORS

Summary

A concept of noncommutative Galois extension is introduced and binary and ternary extensions are chosen. The ternary Clifford algebra is introduced its Clifford analysis and the connection with the Galois extension is indicated. Then we can formulate the binary and ternary Hurwitz-type conditions and we obtain the Dirac operator by these conditions. Hence the binary and ternary Dirac operators can be described in a unified manner.

In the second part of the paper we discuss the binary Hurwitz condition vs. the Dirac and Klein-Gordon operators, the ternary Hurwitz condition vs. the binary Hurwitz condition, and the correspondence between ternary and binary field operators.

Keywords and phrases: Clifford analysis, ternary Clifford analysis, nonion algebra, Galois extension, Dirac and Klein-Gordon field operators, ternary field operators

4. Binary Hurwitz condition vs. the Dirac and Klein-Gordon operators

In this section, we recall the Hurwitz condition for K ($= \mathbb{R}$ or $\mathbb{C}$)-vector space and the Hurwitz problem [5]. Then we can obtain the Dirac operator from the solution.

4.1. Hurwitz condition

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Then we can formulate the Hurwitz condition as follows: A bilinear mapping $\tau : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Hurwitz condition, whenever

$$|\tau(x, y)| = |x|\ |y|,$$

where $|x|$ implies the length of x.

We can prove the following fact: the Hurwitz condition is satisfied, if and only if $n = 1, 2, 4, 8$ and the bilinear mapping is realized by the product of the real number, complex numbers, quaternion numbers, and octonion numbers, respectively. We can derive the Dirac operator from this condition: Setting

$$\tau(x, y) = (x_1 C_1 + x_2 C_2 + \ldots + x_n C_n) y,$$

we have

$${}^t C_i C_j + {}^t C_j C_i = 2\delta_{ij} I \quad (i, j = 1, 2, \ldots, n).$$

After a change of an orthonormal coordinate, setting

$$\gamma_i = {}^t C_1 C_i (i = 2, 3, \ldots, n),$$

we have the Clifford algebra with skew-symmetric generators in the case of $n = 1, 2, 4$:

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij} I \quad (i, j = 2, 3, \ldots, n) \quad ({}^t\gamma_i = -\gamma_i).$$

4.2. Partial and full quantization

We can see that

$$\tau(x, y) = (x_1 I + x_2 \gamma_2 + \ldots + x_n \gamma_n) y.$$

Hence, making the quantization, $x_i \to \partial/\partial x_i$ $(i = 1, 2, \ldots, n)$ we can obtain the Dirac operator by the partial quantization and Fueter operator by the full quantization, respectively [5]:

$$\begin{cases} D\psi = (mI + \gamma_2 \partial/\partial x_2 + \ldots + \gamma_n \partial/\partial x_n)\psi, \\ F\psi = (I_n \partial/\partial x_1 + \gamma_2 \partial/\partial x_2 + \ldots + \gamma_n \partial/\partial x_n)\psi. \end{cases}$$

In the next section we introduce a concept of the ternary Hurwitz condition, obtain the ternary Dirac operator, and give a unified description of the ternary Hurwitz condition and the usual Hurwitz condition. Hereafter this condition is called *binary Hurwitz condition.*

5. Ternary Hurwitz condition vs. the binary Hurwitz condition

In this section we are concerned with the Hurwitz conditions and Hurwitz problems.

5.1. Ternary/binary character

We begin with a concept of ternary character.

Definition 5. Let A be a matrix algebra over the field K. Then a mapping $\rho : A \to K$ is said to be *of ternary character iff*

$$\rho(xyz) = \rho(x)\rho(y)\rho(z)$$

holds. Here we notice that we do not necessarily assume that $\rho(xy) = \rho(x)\rho(y)$. Especially when the latter relation is satisfied, is called *of character of binary type*, of *binary character*, or *of trivial character*.

We notice the following fact: Every matrix algebra has a trivial character:

$$\rho(x) = \det\, x (x \in A).$$

We can propose the following problem:

Problem 1. Can we find another type of character?

5.2. Ternary/binary Hurwitz condition

Now we shall introduce a concept of ternary Hurwitz condition:

1) We assume that A is a ternary involutive algebra. Namely it has a ternary involution: The automorphism $x \to x^*$ satisfying $x^{***} = x$. When it has the following type of ternary character, we say that the algebra satisfies the ternary Hurwitz condition:

$$\rho(x)\rho(x^*)\rho(x^{**}) = \rho(x)^3.$$

2) In a similar manner we assume that A has the following type of binary involution. We say that the algebra satisfies the binary Hurwitz condition when:

$$\rho(x)\rho(x^*) = \rho(x)^2$$

where $x \to x^*$ is the binary involution.

Proposition 2. *We can obtain the characters on the nonion algebra and its binary extension as follows:*

1) *We have the character of the following form with respect to the linear elements* $\{\theta_1, \theta_2, \theta_3\}_X (X = Q, \overline{Q}, R)$*:*

$$\rho(\theta) = \det\, \{\theta_1, \theta_2, \theta_3\}_X = \theta_1^3 + \theta_2^3 + \theta_3^3 - 3\theta_1\theta_2\theta_3.$$

2) *We have the characters of the linear elements on* $\tilde{N}$ *with respect to the coordinates* $\{\theta_1, \theta_2, \theta_3\}_X (X = Q, \overline{Q}, R, T_4)$*:*

$$\rho(\theta) = -\det\, \{\theta_1, \theta_2, \theta_3\}_X (X = T_4) = -(\theta_1^3 + \theta_2^3 + \theta_3^3 - 3\theta_1\theta_2\theta_3).$$

The proofs are direct calculations which may be omitted.

5.3. Ternary Hurwitz problem

As for the Hurwitz condition we can propose the following problem:

Problem 2. Determine the algebra which admits a ternary Hurwitz character and determine all the characters.

We can prove the following theorem:

Theorem 1. 1) *The binary Hurwitz problem can be solved in the following manner: Choosing A – one of the algebras of the real numbers, complex numbers, quaternionic numbers, the character $\rho : A \to K$ can be expressed as follows:*

$$\rho(z) = |z|^z \quad (n = 1, 2, 4).$$

2) *Ternary Hurwitz problem can be solved on the nonion algebra. Let N be the nonion algebra and let $\{\theta_1, \theta_2, \theta_3\}_X (X = Q, \overline{Q}, R)$ be a linear element. Then the one of the ternary Hurwitz characters can be written as follows:*

$$\rho(\theta) = \det\, \{\theta_1, \theta_2, \theta_3\}_X = \theta_1^3 + \theta_2^3 + \theta_3^3 - 3\theta_1\theta_2\theta_3.$$

Proof. Ad 1). Choosing a complex number $z = x + iy$, we consider the matrix representation of complex numbers and their conjugate numbers as follows:

$$z(\theta) = \begin{pmatrix} \theta_1 & -\theta_2 \\ \theta_2 & \theta_1 \end{pmatrix}, \qquad z^*(\theta) = \begin{pmatrix} \theta_1 & \theta_2 \\ -\theta_2 & \theta_1 \end{pmatrix}.$$

Then the following character satisfies the binary Hurwitz condition:

$$\rho(zw) = \rho(z)\rho(w)$$

which is nothing but $|zw|^2 = |z|^2|w|^2$:

$$\rho(z) = \rho(z^*) = \begin{vmatrix} \theta_1 & -\theta_2 \\ \theta_2 & \theta_1 \end{vmatrix} = \left(\theta_1^2 + \theta_2^2\right).$$

Ad 2). The proof of 2) is a direct consequence of Proposition 2. We choose a quaternion number $z = \theta_1 + i\theta_2 + j\theta_3 + k\theta_k$ and we consider the matrix representation:

$$z(\theta) = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ -\theta_2 & \theta_1 & \theta_4 & -\theta_3 \\ -\theta_3 & -\theta_4 & \theta_1 & \theta_2 \\ -\theta_4 & \theta_3 & -\theta_2 & \theta_1 \end{pmatrix}, \qquad z^*(\theta) = \begin{pmatrix} \theta_1 & -\theta_2 & -\theta_3 & -\theta_4 \\ -\theta_2 & \theta_1 & -\theta_4 & \theta_3 \\ \theta_3 & \theta_4 & \theta_1 & -\theta_2 \\ \theta_4 & -\theta_3 & \theta_2 & \theta_1 \end{pmatrix}.$$

Then the following character satisfies the binary Hurwitz condition

$$\rho(zw) = \rho(z)\rho(w)$$

which is nothing but $|zw|^4 = |z|^4|w|^4$:

$$\rho(z) = \rho(z^*) = \begin{vmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ -\theta_2 & \theta_1 & \theta_4 & -\theta_3 \\ \theta_3 & \theta_4 & \theta_1 & \theta_2 \\ \theta_4 & -\theta_3 & -\theta_2 & \theta_1 \end{vmatrix} \quad \left(= (\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2)^2\right).$$

6. The correspondence between ternary and binary field operators

We are concerned with the relationship between binary and ternary Dirac operators. This relation will be very important in considering the quark confinement.

6.1. Binary repetition of ternary extensions and ternary repetition of binary extensions

In fact, proton or neutron constitutes three quarks. Hence quarks can be described in terms of ternary Dirac operators. On the other hand, we can describe proton or neutron in terms of the binary Dirac operators. Therefore we have to make clear the relation between these equations. We shall describe this relationship in terms of Galois extensions. Here we assume that a pair of repetition of binary and ternary extensions is given:

$$\begin{cases} \tilde{A} = \sqrt[2]{I_n}[A], \quad A = \sqrt[3]{I_n}[A_0], \\ \\ \tilde{A} = \sqrt[3]{I_n}[B], \quad B = \sqrt[2]{I_n}[B_0]. \end{cases}$$

Then we can define the binary and ternary characters from the extensions and we can obtain kinds of Dirac operators. By this we can discuss the relationship between the baryons and the triple quarks.

We can obtain the corresponding Dirac operators in the following different ways (cf. Fig. 5).

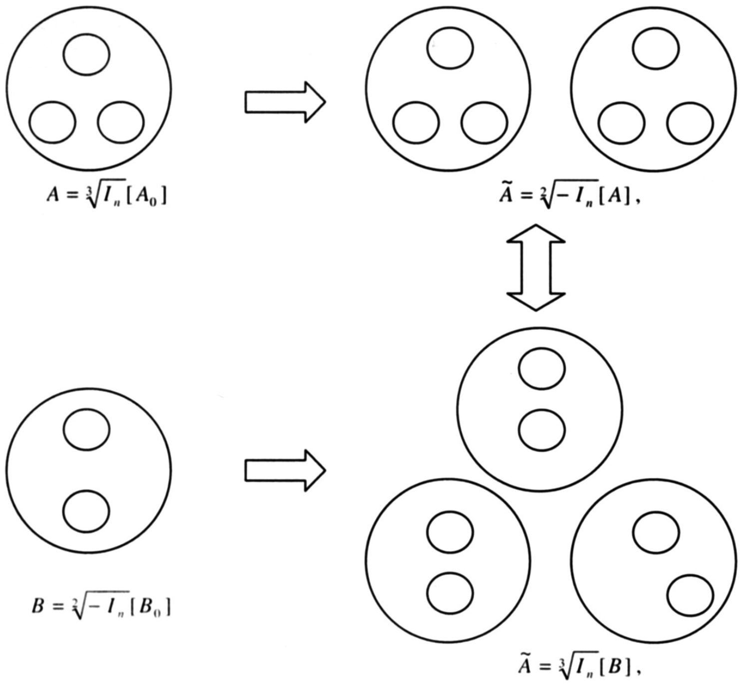

Fig. 5: A pair of repetition of binary and ternary extensions, e.g. in the case of proton or neutron, which: a) constitutes three quarks, b) can be described in terms of ternary Dirac operators.

a) The case of $A = \sqrt[3]{I_n}[A_0]$, $\tilde{A} = \sqrt[2]{I_n}[A]$. Using basic constructions

$$\sqrt[2]{-1} \Leftrightarrow \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}, \qquad \sqrt[3]{1} \Leftrightarrow \begin{pmatrix} y_1 & y_2 & y_3 \\ y_3 & y_1 & y_2 \\ y_2 & y_3 & y_1 \end{pmatrix},$$

we make the composition of the extensions:

$$\sqrt[2]{-1} \otimes \sqrt[3]{1} \Leftrightarrow \left(\begin{array}{ccc|ccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ y_3 & y_1 & y_2 & y_6 & y_4 & y_5 \\ y_2 & y_3 & y_1 & y_5 & y_6 & y_4 \\ \hline -y_4 & -y_5 & -y_6 & y_1 & y_2 & y_3 \\ -y_6 & -y_4 & -y_5 & y_3 & y_1 & y_2 \\ -y_5 & -y_6 & -y_4 & y_2 & y_3 & y_1 \end{array}\right).$$

Hence we obtain the following Dirac operator:

$$\tilde{D}_1 = T_1 \frac{\partial}{\partial y_1} + T_2 \frac{\partial}{\partial y_2} + T_3 \frac{\partial}{\partial y_3} + T_4 \frac{\partial}{\partial y_4} + T_5 \frac{\partial}{\partial y_5} + T_6 \frac{\partial}{\partial y_6},$$

where

$$T_1 = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & 0_3 & \\ 0 & 0 & 1 & & & \\ \hline & & & 1 & 0 & 0 \\ & 0_3 & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right), \quad T_2 = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & 0_3 & \\ 1 & 0 & 0 & & & \\ \hline & & & 0 & 1 & 0 \\ & 0_3 & & 0 & 0 & 1 \\ & & & 1 & 0 & 0 \end{array}\right), \quad T_3 = \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & & & \\ 1 & 0 & 0 & & 0_3 & \\ 0 & 1 & 0 & & & \\ \hline & & & 0 & 0 & 1 \\ & 0_3 & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \end{array}\right),$$

$$T_4 = \left(\begin{array}{ccc|ccc} & & & 1 & 0 & 0 \\ & 0_3 & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & & & \\ 0 & -1 & 0 & & 0_3 & \\ 0 & 0 & -1 & & & \end{array}\right), \quad T_5 = \left(\begin{array}{ccc|ccc} & & & 0 & 1 & 0 \\ & 0_3 & & 0 & 0 & 1 \\ & & & 1 & 0 & 0 \\ \hline 0 & -1 & 0 & & & \\ 0 & 0 & -1 & & 0_3 & \\ -1 & 0 & 0 & & & \end{array}\right), \quad T_6 = \left(\begin{array}{ccc|ccc} & & & 0 & 0 & 1 \\ & 0_3 & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ \hline 0 & 0 & -1 & & & \\ -1 & 0 & 0 & & 0_3 & \\ 0 & -1 & 0 & & & \end{array}\right).$$

Consequently we arrive at the binary Dirac operators

$$\begin{cases} D_b^{(1)} = T_1 \dfrac{\partial}{\partial y_1} + T_4 \dfrac{\partial}{\partial y_4}, \\[2ex] D_b^{(1)*} = T_1^* \dfrac{\partial}{\partial y_4} + T_4^* \dfrac{\partial}{\partial y_4}; \end{cases} \qquad \begin{cases} D_b^{(2)} = T_2 \dfrac{\partial}{\partial y_2} + T_5 \dfrac{\partial}{\partial y_5}, \\[2ex] D_b^{(2)*} = T_2^* \dfrac{\partial}{\partial y_2} + T_4^* \dfrac{\partial}{\partial y_5}; \end{cases}$$

$$\begin{cases} D_b^{(3)} = T_3 \dfrac{\partial}{\partial y_3} + T_6 \dfrac{\partial}{\partial y_6}, \\[2ex] D_b^{(3)*} = T_3^* \dfrac{\partial}{\partial y_3} + T_4^* \dfrac{\partial}{\partial y_6}. \end{cases}$$

b) For the case of $\tilde{A} = \sqrt[2]{I_n}[B]$, $B = \sqrt[3]{I_n}[B_0]$, we have in a completely analogous manner

$$\sqrt[3]{1} \Leftrightarrow \begin{pmatrix} y_1 & y_2 & y_3 \\ y_3 & y_1 & y_2 \\ y_2 & y_3 & y_1 \end{pmatrix}, \qquad \sqrt[2]{1} \Leftrightarrow \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix},$$

Hence we get the following realization:

$$\sqrt[3]{1} \otimes \sqrt[2]{-1} \Leftrightarrow \left(\begin{array}{cc|cc|cc} x_1 & x_2 & y_1 & y_2 & z_1 & z_2 \\ -x_2 & x_1 & -y_2 & y_1 & -z_2 & z_1 \\ \hline z_1 & z_2 & x_1 & x_2 & y_1 & y_2 \\ -z_2 & z_1 & -x_2 & x_1 & -y_2 & y_1 \\ \hline y_1 & y_2 & z_1 & z_2 & x_1 & x_2 \\ -y_2 & y_1 & -z_2 & z_1 & -x_2 & x_1 \end{array}\right).$$

Therefore we obtain the following Dirac operator:

$$\tilde{D} = S_1 \frac{\partial}{\partial y_1} + S_2 \frac{\partial}{\partial y_2} + S_3 \frac{\partial}{\partial y_3} S_4 \frac{\partial}{\partial y_4} + S_5 \frac{\partial}{\partial y_5} + S_6 \frac{\partial}{\partial y_6},$$

where

$$S_1 = \left(\begin{array}{c|c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0_2 & 0_2 \\ \hline 0_2 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0_2 \\ \hline 0_2 & 0_2 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array}\right), \qquad S_2 = \left(\begin{array}{c|c|c} 0_2 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0_2 \\ \hline 0_2 & 0_2 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \hline \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0_2 & 0_2 \end{array}\right),$$

$$S_3 = \left(\begin{array}{c|c|c} 0_2 & 0_2 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \hline \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0_2 & 0_2 \\ \hline 0_2 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0_2 \end{array}\right), \qquad S_4 = \left(\begin{array}{c|c|c} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0_2 & 0_2 \\ \hline 0_2 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0_2 \\ \hline 0_2 & 0_2 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right),$$

$$S_5 = \left(\begin{array}{cc|cc|cc} & & 0 & 1 & & \\ \multicolumn{2}{c|}{0_2} & & & \multicolumn{2}{c}{0_2} \\ & & -1 & 0 & & \\ \hline & & & & 0 & 1 \\ \multicolumn{2}{c|}{0_2} & \multicolumn{2}{c|}{0_2} & & \\ & & & & -1 & 0 \\ \hline 0 & 1 & & & & \\ & & \multicolumn{2}{c|}{0_2} & \multicolumn{2}{c}{0_2} \\ -1 & 0 & & & & \end{array}\right), \qquad S_6 = \left(\begin{array}{cc|cc|cc} & & & & 0 & 1 \\ \multicolumn{2}{c|}{0_2} & \multicolumn{2}{c|}{0_2} & & \\ & & & & -1 & 0 \\ \hline 0 & 1 & & & & \\ & & \multicolumn{2}{c|}{0_2} & \multicolumn{2}{c}{0_2} \\ -1 & 0 & & & & \\ \hline & & 0 & 1 & & \\ \multicolumn{2}{c|}{0_2} & & & \multicolumn{2}{c}{0_2} \\ & & -1 & 0 & & \end{array}\right),$$

Consequently we arrive at the ternary Dirac operators

$$\begin{cases} D_t^{(1)} = S_1\dfrac{\partial}{\partial y_1} + S_2\dfrac{\partial}{\partial y_2} + S_3\dfrac{\partial}{\partial y_3}, \\[2ex] D_t^{(1)} = S_1\dfrac{\partial}{\partial y_1} + \mathrm{j}S_2\dfrac{\partial}{\partial y_2} + \mathrm{j}^2 S_3\dfrac{\partial}{\partial y_3}, \\[2ex] D_t^{(1)} = S_1\dfrac{\partial}{\partial y_1} + \mathrm{j}^2 S_2\dfrac{\partial}{\partial y_2} + \mathrm{j}S_3\dfrac{\partial}{\partial y_3}, \end{cases}$$

$$\begin{cases} D_t^{(2)} = S_4\dfrac{\partial}{\partial y_1} + S_5\dfrac{\partial}{\partial y_2} + S_6\dfrac{\partial}{\partial y_3}, \\[2ex] D_t^{(2)} = S_4\dfrac{\partial}{\partial y_1} + \mathrm{j}S_5\dfrac{\partial}{\partial y_2} + \mathrm{j}^2 S_6\dfrac{\partial}{\partial y_3}, \\[2ex] D_t^{(2)} = S_4\dfrac{\partial}{\partial y_1} + \mathrm{j}^2 S_5\dfrac{\partial}{\partial y_2} + \mathrm{j}S_6\dfrac{\partial}{\partial y_3}. \end{cases}$$

Here we notice the following identification:

$$\left(\begin{array}{cc|cc|cc} x_1 & x_2 & y_1 & y_2 & z_1 & z_2 \\ -x_2 & x_1 & -y_2 & y_1 & -z_2 & z_1 \\ \hline z_1 & z_2 & x_1 & x_2 & y_1 & y_2 \\ -z_2 & z_1 & -x_2 & x_1 & -y_2 & y_1 \\ \hline y_1 & y_2 & z_1 & z_2 & x_1 & x_2 \\ -y_2 & y_1 & -z_2 & z_1 & -x_2 & x_1 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|ccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ y_3 & y_1 & y_2 & y_6 & y_4 & y_5 \\ y_2 & y_3 & y_1 & y_5 & y_6 & y_4 \\ \hline -y_4 & -y_5 & -y_6 & y_1 & y_2 & y_3 \\ -y_6 & -y_4 & -y_5 & y_3 & y_1 & y_2 \\ -y_5 & -y_6 & -y_4 & y_2 & y_3 & y_1 \end{array}\right).$$

6.2. The double extension of the binary and ternary Galois extensions, and the related double fibering

Discussions of the subsection 6.1 imply

Theorem 2. *When we have the double extension of the binary and ternary Galois extensions* $\tilde{A} = \sqrt[2]{-1}[A]$, $\tilde{A} = \sqrt[3]{1}[B]$*:*

$$
\begin{array}{ccc}
 & \tilde{A} & \\
\tau \swarrow & & \searrow \eta \\
A & & B
\end{array}
$$

the following assertions hold:

1) *We have the universal Dirac operators for the both extensions and they are equivalent each other as a consequence of the Hurwitz condition*

$$\rho : \tilde{A} \to K.$$

2) *We have the following double fibering. We have the Dirac operators from the characters $\rho_A : A \to K$ and $\rho_B : B \to K$, namely:*

$$
\begin{gathered}
\tilde{D}_1 = T_1 \frac{\partial}{\partial y_1} + T_2 \frac{\partial}{\partial y_2} + T_3 \frac{\partial}{\partial y_3} + T_4 \frac{\partial}{\partial y_4} + T_5 \frac{\partial}{\partial y_5} + T_6 \frac{\partial}{\partial y_6} \\
\left(\tilde{D}_2 = S_1 \frac{\partial}{\partial y_1} + S_2 \frac{\partial}{\partial y_2} + S_3 \frac{\partial}{\partial y_3} + S_4 \frac{\partial}{\partial y_4} + S_5 \frac{\partial}{\partial y_5} + S_6 \frac{\partial}{\partial y_6} \right)
\end{gathered}
$$

$$
\rho_A = \tau^*(\rho_{\tilde{A}}) \swarrow \qquad\qquad \searrow \rho_B = \eta^*(\rho_{\tilde{A}})
$$

$$
\begin{cases}
D_t = S_1 \dfrac{\partial}{\partial y_1} + S_2 \dfrac{\partial}{\partial y_2} + S_3 \dfrac{\partial}{\partial y_3}, \\[2ex]
D_t = S_1 \dfrac{\partial}{\partial y_1} + \mathrm{j} S_2 \dfrac{\partial}{\partial y_2} + \mathrm{j}^2 S_3 \dfrac{\partial}{\partial y_3}, \\[2ex]
D_t = S_1 \dfrac{\partial}{\partial y_1} + \mathrm{j}^2 S_2 \dfrac{\partial}{\partial y_2} + j S_3 \dfrac{\partial}{\partial y_3};
\end{cases}
\qquad
\begin{cases}
D_b = T_1 \dfrac{\partial}{\partial y_1} + T_2 \dfrac{\partial}{\partial y_2} + T_3 \dfrac{\partial}{\partial y_3}, \\[2ex]
D_b^* = T_4 \dfrac{\partial}{\partial y_4} + T_5 \dfrac{\partial}{\partial y_5} + T_6 \dfrac{\partial}{\partial y_6}.
\end{cases}
$$

In this manner we can give the equivalence between the ternary particles and the binary particles.

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BINARNA I TERNARNA ANALIZA CLIFFORDA A NIEPRZEMIENNE ROZSZERZENIA GALOIS II

ODPOWIEDNIOŚĆ MIĘDZY TERNARNYMI I BINARNYMI OPERATORAMI POLA

S t r e s z c z e n i e

Wprowadzono pomysł nieprzemiennego rozszerzenia Galois przy wyborze rozszerzeń binarnych i ternarnych. Wprowadzono ternarną algebrę Clifforda oraz naszkicowano odpowiednią analizę Clifforda i związek z rozszerzeniem Galois. W konsekwencji potrafimy sformułować binarne i ternarne warunki typu Hurwitza i uzyskać z tych warunków operator Diraca. Tak więc operatory Diraca, binarny i ternarny, dadzą się jednolicie scharakteryzować.

W drugiej części pracy omawiamy binarny warunek Hurwitza w odniesieniu do operatorów Diraca i Kleina-Gordona, ternarny warunek Hurwitza w odniesieniu do binarnego warunku Hurwitza oraz odpowiedniość między operatorami pola: ternarnym i binarnym.

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