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DE LA SOCIÉTÉ DES SCIENCES
ET DES LETTRES DE ŁÓDŹ

SÉRIE:
RECHERCHES SUR LES DÉFORMATIONS

Volume LXVI, no. 2

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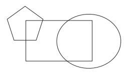

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Professors Julian Ławrynowicz and Leszek Wojtczak
during the ceremony of renewing of their doctorates (1964) after 50 years

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Contribution to the jubilee volume, dedicated to Professors J. Ławrynowicz and L. Wojtczak

Aleksandr Bakhtin, Inna Dvorak, and Iryna Denega

SEPARATING TRANSFORMATION AND EXTREMAL DECOMPOSITION OF THE COMPLEX PLANE

Summary

The paper is devoted to extremal problems of the geometric function theory of complex variable associated with estimates of functionals defined on systems of non-overlapping domains. In particular, focus of investigation is well-known Dubinin's problem about extremal decomposition of the complex plane.

Keywords and phrases: inner radius, non-overlapping domains, "free" poles, n-radial system of points, Dubinin problem, inequalities, quadratic differential

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ be a set of natural and real numbers, respectively, $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be a Riemann sphere and $\mathbb{R}^+ = (0, \infty)$.

A finite set of arbitrary domains $\{B_k\}_{k=1}^n$, $n \in \mathbb{N}$, $n \geqslant 2$ such, as $B_k \subset \overline{\mathbb{C}}$, $B_k \cap B_m = \emptyset$, $k \neq m$, $k, m = \overline{1,n}$ is called *a system of non-overlapping domains.*

Let

$$r(B,a) = \begin{cases} \exp(\lim\limits_{z\to a}(g_B(z,a) + \log|z-a|)), & a \neq \infty \\ \exp(\lim\limits_{z\to a}(g_B(z,a) - \log|z|)), & a = \infty \end{cases}$$

be an inner radius of the domain $B \subset \overline{\mathbb{C}}$ with respect to the point $a \in B$, $g_B(z,a)$ is a Green's function for the domain B. Inner radius is a generalization of conformal radius for multiply connected domains (see [1–3]).

Let $n \in \mathbb{N}$. A set of points

$$A_n := \left\{a_k \in \mathbb{C} : \ k = \overline{1,n}\right\},$$

is called *n-radial* system, if

$$|a_k| \in \mathbb{R}^+, \quad k = \overline{1,n}, \quad \text{and} \quad 0 = \arg a_1 < \arg a_2 < \ldots < \arg a_n < 2\pi.$$

Denote

$$\alpha_k := \frac{1}{\pi} \arg \frac{a_{k+1}}{a_k}, \ \alpha_{n+1} := \alpha_1, \ k = \overline{1,n}.$$

Consider next problem.

Problem 1. Show that the maximum of the product

$$J_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

where B_0, B_1, B_2,...,B_n, $n \geqslant 2$ are pairwise non-overlapping domains in $\overline{\mathbb{C}}$, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1,n}$ and $\gamma > 0$, achieved for some configuration of the domains B_k and points a_k, $k = \overline{0,n}$, which are having n-fold symmetry.

This problem has a solution only if $\gamma \leqslant n$, as soon as $\gamma = n + \varepsilon$, $\varepsilon > 0$, the Problem 1 has no solution.

Problem 1 was formulated in 1994 [4], and then repeated in 2009 [5]. Currently it is not solved in general, known only special results.

In 1988 in Dubinin's work [6] this problem was solved for $\gamma = 1$ and $n \geqslant 2$, and from the method of this work it implies that the result is true and for $0 < \gamma < 1$.

In 1996 L. V. Kovalev [7] got the solution to this problem with some restrictions on the geometry location of sets of points on the unit circle and, namely, for $n \geqslant 5$ and subclass points systems satisfying condition

$$0 < \alpha_k \leqslant 2/\sqrt{\gamma}, \ k = \overline{1,n}.$$

It is clear that these conditions are sufficiently stringent conditions, significantly narrowing the set of feasible configurations. It should be noted that the Kovalev's result is interesting not only by itself, but the method of study is important too.

In 2003 in the paper of G. V. Kuzmina [8] in the case of simply connected domains, this problem has also been studied for $\gamma \in (0, 1]$ by another method.

In 2008 A. K. Bakhtin [9, p. 255] complemented the ideas and methods of previous works and thus he showed, that the result of V. N. Dubinin holds for arbitrary $\gamma \in \mathbb{R}^+$, but since some number $n_0(\gamma)$.

It should be noted in the case $\gamma > 1$ method, developed in the paper of V. N. Dubinin [6], can not be applied.

In 2013 Y. V. Zabolotnii [10] got the solution of Problem 1 for $n \geqslant 2$ and $0 < \gamma \leqslant \sqrt[4]{n}$, and for $0 < \gamma \leqslant n^{\alpha}$, where

$$\frac{1}{3} < \alpha < \frac{2}{3} \quad \text{since some number} \quad n \geqslant e^{\frac{1}{\left(\frac{2}{3} - \alpha\right)^2}}.$$

2. Main results

We got the following result:

Theorem 1. *Let $n \in \mathbb{N}$, $n \geqslant 12$, $\gamma \in (0, \gamma_n]$, $\gamma_n = n^{0,45}$. Then for any n-radial system of points $A_n = \{a_k\}_{k=1}^{n}$ such that $|a_k| = 1$ and any system of non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $a_0 = 0 \in B_0$, $(k = \overline{1,n})$, we have inequality*

$$r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \left(\frac{4}{n}\right)^{n} \cdot \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \cdot \left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}},$$

where equality holds if a_k and B_k, $k = \overline{0,n}$, are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n + \gamma}{w^2(w^n-1)^2}\, dw^2.$$

Proof. L. V. Kovalev [7] proved the hypothesis of V. N. Dubinin if $n \geqslant 5$ and $\alpha_k \sqrt{\gamma} \leqslant 2$, $k = \overline{1,n}$. We must investigate the case $\alpha_0 \sqrt{\gamma} \geqslant 2$, $\alpha_0 = \max_k \alpha_k$. According to the method of work [9, theorem 5.2.3] we have

$$J_n(\gamma) = \prod_{k=1}^{n} \left[r(B_0,0) r(B_k,a_k)\right]^{\frac{\gamma}{n}} \left[\prod_{k=1}^{n} r(B_k,a_k)\right]^{1-\frac{\gamma}{n}}.$$

From theorem of M. A. Lavrent'ev [1] next inequality holds

$$r(B_0,0) r(B_k,a_k) \leqslant |a_k|^2.$$

Further from [9, theorem 5.1.1] we have

$$\prod_{k=1}^{n} r(B_k,a_k) \leqslant 2^n \prod_{k=1}^{n} \alpha_k.$$

Since $\sum\limits_{k=1}^{n} \alpha_k = 2$, we apply the Cauchy inequality between the geometric mean and the arithmetic mean. So,

$$J_n(\gamma) \leqslant \left[2^n \alpha_0 \left(\frac{2-\alpha_0}{n-1}\right)^{n-1}\right]^{1-\frac{\gamma}{n}},$$

where $\alpha_0 = \max_k \alpha_k$, $\alpha_0 \geqslant \frac{2}{\sqrt{\gamma}}$. On the other hand it is known [7, 9], that

$$J_n^0(\gamma) = \left(\frac{4}{n}\right)^{n} \cdot \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \cdot \left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}.$$

$J_n^0(\gamma)$ was first obtained in paper [6] if $\gamma = 1$, for any γ in paper [7], then in another form in monograph [9, p. 257].

Consider next value $O_n(\gamma) = J_n(\gamma)/J_n^0(\gamma)$ when $\alpha_0\sqrt{\gamma} \geqslant 2$. Performing simple transformations we obtain

$$O_n(\gamma) \leqslant \left[\frac{n}{4}\right]^{\gamma+1} \cdot \left[1-\frac{1}{\sqrt{\gamma}}\right]^{n-1-\gamma\frac{n-1}{n}} \cdot \left(\frac{n}{\gamma}\right)^{\frac{\gamma}{n}} \cdot \left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}} \times$$
$$\times \left(\frac{1+\frac{\sqrt{\gamma}}{n}}{1-\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}} \cdot \left(\frac{4}{\sqrt{\gamma}}\right)^{1-\frac{\gamma}{n}} \cdot \left(\frac{n}{n-1}\right)^{n-1-\gamma\frac{n-1}{n}}.$$

In an analogous way as in papers [9, p.255-259], [10, 11], we investigate evaluation of each factor of $O_n(\gamma)$ according to standard scheme. And thus we showed that $J_n(\gamma) < J_n^0(\gamma)$ if $\gamma \in (1, n^{0,45}]$, $\alpha_0\sqrt{\gamma} \geqslant 2$, $n \geqslant 12$. Hence it follows, that we do not have extremal configurations for these values of the parameters. Further, we analyze the case $\alpha_0\sqrt{\gamma} < 2$. Using the results [9–11], we obtain

$$J_n(\gamma) \leqslant \gamma^{-\frac{n}{2}} \left[\prod_{k=1}^{n} P\left(\alpha_k\sqrt{\gamma}\right)\right]^{\frac{1}{2}},$$

where

$$P(x) = 2^{x^2+6} \cdot x^{x^2+2} \cdot (2-x)^{-\frac{1}{2}(2-x)^2} \cdot (2+x)^{-\frac{1}{2}(2+x)^2}, \quad x \in (0,2].$$

Applying the reasoning of paper [7], we have Theorem 1. A sign of equality is verified directly. Theorem 1 is thus proved.

We also obtained another most strong result for Problem 1. Let

$$F_\delta(x) = 2^{x^2+6} \cdot x^{x^2+2-2\delta} \cdot (2-x)^{-\frac{1}{2}(2-x)^2}(2+x)^{-\frac{1}{2}(2+x)^2},$$
$$x \in (0,2], \quad 0 \leqslant \delta \leqslant 0,7.$$

Theorem 2. *Let $n \in \mathbb{N}$, $n \geqslant 5$, $\gamma \in (0, \gamma_0]$, $\gamma_0 = \sqrt[4]{n}$, $0 \leqslant \delta \leqslant 0,7$. Then for any n-radial system of points $A_n = \{a_k\}_{k=1}^n$ such that $|a_k| = 1$ and any system of non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, $a_0 = 0$, we have inequality*

$$r^\gamma\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \gamma^{-\frac{\delta \cdot n}{2}} \cdot \left(\prod_{k=1}^{n} \alpha_k\right)^{\delta} \cdot \left[F_\delta\left(\frac{2}{n}\sqrt{\gamma}\right)\right]^{\frac{n}{2}},$$

where equality holds in the same case as in Theorem 1.

Proof. Method of proof of this theorem is based on using of separating transformation [4, 6] and ideas of works [7, 9, 12, 13]. In an analogous way as in papers [11–13] we obtain next inequality

$$J_n(\gamma) \leqslant \gamma^{-\frac{\delta n}{2}} \left(\prod_{k=1}^{n} \alpha_k\right)^{\delta} \left[\prod_{k=1}^{n} F_\delta\left(\alpha_k\sqrt{\gamma}\right)\right]^{1/2}, \ \delta \in [0; 0,7]. \tag{1}$$

Consider the functional

$$\tilde{J}_n(\gamma) = \gamma^{\frac{\delta n}{2}} \left(\prod_{k=1}^{n} \alpha_k\right)^{-\delta} J_n(\gamma).$$

It follows from (1) that

$$\tilde{J}_n(\gamma) \leqslant \left[\prod_{k=1}^{n} F_\delta\left(\alpha_k\sqrt{\gamma}\right)\right]^{\frac{1}{2}}.$$

Further, consider an extremal problem

$$\prod_{k=1}^{n} F_\delta(x_k) \longrightarrow \max, \quad \sum_{k=1}^{n} x_k = 2\sqrt{\gamma}, \quad x_k = \alpha_k\sqrt{\gamma}$$

$$0 < x_k \leqslant 2, \quad 0 \leqslant \delta \leqslant 0,7.$$

Let $\Psi_\delta(x) = \ln\left(F_\delta(x)\right)$. And let $X^{(0)} = \left\{x_k^{(0)}\right\}_{k=1}^{n}$ be an arbitrary extremal point of this problem. It follows from the paper [7] that: if $0 < x_k^{(0)} < x_j^{(0)} < 2$, $k \neq j$, than $\Psi'_\delta(x_k^{(0)}) = \Psi'_\delta(x_j^{(0)})$, where $k, j = \overline{1,n}$, $k \neq j$, $0 \leqslant \delta \leqslant 0,7$,

$$\Psi'_\delta(x) = 2x\ln(2x) + (2-x)\ln(2-x) - (2+x)\ln(2+x) + \frac{2}{x} - \frac{2\delta}{x}$$

(see Fig. 1).

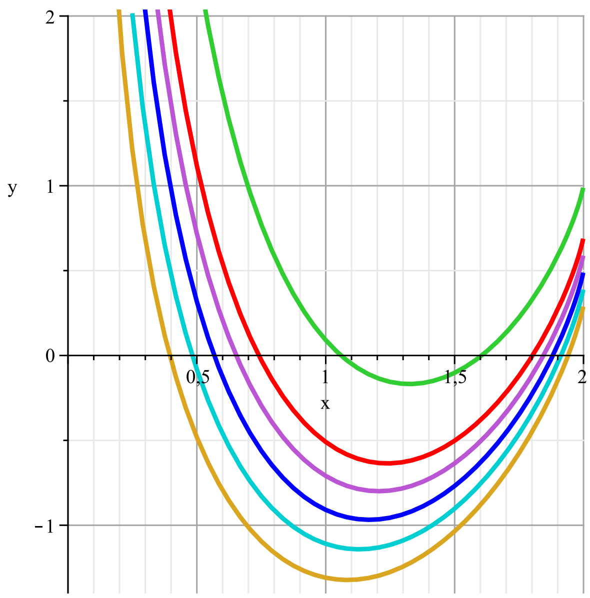

Fig. 1: Graph of the function $y = \Psi'_\delta(x)$

.

We will show that the following assertion is true

$$x_1^{(0)} = x_2^{(0)} = \ldots = x_n^{(0)}.$$

Let $\sigma_1 := \sigma_1(\delta,\gamma) = \min\limits_{1\leqslant k\leqslant n} x_k^{(0)}(\delta,\gamma)$, $\sigma_0 := \sigma_0(\delta,\gamma) = \max\limits_{1\leqslant k\leqslant n} x_k^{(0)}(\delta,\gamma)$, $\sigma_1 \leqslant \sigma_k \leqslant \sigma_0$, $k=\overline{1,n}$, $0\leqslant\delta\leqslant 0,7$, $\gamma\in(0;\sqrt[4]{n}]$.

A function $\Psi''_\delta(x)$ is strictly increasing on $(0,2)$ for each fixed δ. Thus

$$sign\Psi''_\delta(x) \equiv sign(x - x_0(\delta,\gamma)).$$

It is not difficult to see if $\sigma_0 \leqslant x_0(\delta,\gamma)$ then from the conditions of the problem we obtain $x_1^{(0)} = \ldots = x_n^{(0)}$.

Assume $x_0(\delta,\gamma) < \sigma_0 \leqslant 1,81$. Then

$$\sigma_1 \leqslant \frac{1}{n-1}\sum_{k=1}^{n-1} x_k^{(0)} = \frac{2\sqrt{\gamma}-\sigma_0}{n-1} \leqslant \frac{2\sqrt[8]{n}-\sigma_0}{n-1} \leqslant \frac{2\sqrt[8]{5}-\sigma_0}{4}.$$

Therefore for $n\geqslant 5$, $\gamma\in(0;\sqrt[4]{n}]$, $0\leqslant\delta\leqslant 0,7$, inequality holds

$$\sigma_1 \leqslant (2\sqrt[8]{n} - x_0(\delta,\gamma))/4 \leqslant (2,445689 - 1,08441)/4 < 0,340320.$$

Thus

$$\Psi'_\delta(\sigma_1) > \Psi'_\delta(0,340320) > \Psi'_{0,7}(0,340320) = 0,352079 >$$
$$> 0,350102 = \Psi'_0(1,81) \geqslant \Psi'_\delta(\sigma_0).$$

Hence $\sigma_0 \notin (x_0(\delta,\gamma);\, 1,81]$.

Let $1,81 < \sigma_0 \leqslant 2$. Then for $n\geqslant 5$, $\gamma\in(0;\sqrt[4]{n}]$, $0\leqslant\delta\leqslant 0,7$, we have $\sigma_1 \leqslant (2\sqrt[8]{n} - 1,81)/4 < 0,158922$. That is

$$\Psi'_\delta(\sigma_1) > \Psi'_\delta(0,158922) > \Psi'_{0,7}(0,158922) = 2,873304 > 1 = \Psi'_0(2) \geqslant \Psi'_\delta(\sigma_0).$$

So, $\sigma_0 \notin (x_0(\delta,\gamma);\, 2]$. Therefore, extremal set $\left\{x_k^{(0)}\right\}_{k=1}^{n}$ is possible only if

$$x_1^{(0)} = x_2^{(0)} = \ldots = x_n^{(0)}.$$

From the foregoing the following relation holds

$$\prod_{k=1}^{n} F_\delta\left(\alpha_k\sqrt{\gamma}\right) \leqslant \left[F_\delta\left(\frac{2}{n}\sqrt{\gamma}\right)\right]^n.$$

A sign of equality is verified directly. Theorem 2 is thus proved.

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References

[1] M. A. Lavrent'ev, *On the theory of conformal mappings*, Tr. Sci. Inst An USSR **5** (1934), 159–245. (Russian)

[2] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, 26, Amer. Math. Soc., Providence, R.I., 1969.

[3] V. K. Hayman, *Multivalent Functions*, Cambridge University Press, Cambridge 1958.

[4] V. N. Dubinin, *Symmetrization method in geometric function theory of complex variables*, Successes Mat. Science. **49**, no. 1 (295) (1994), 3–76 (Russian); transl: Russian Math. Surveys **49**, no.1 (1994), 1–79.

[5] V. N. Dubinin, *Capasities of condensers and symmetrization in geometric function theory of complex variables*, Vladivostok: Dal'nayka **390** (2009). (Russian)

[6] V. N. Dubinin, *Separating transformation of domains and problems on extremal decomposition*, Notes scientific. sem. Leningr. Dep. of Math. Inst. AN USSR. **168** (1988), 48–66 (Russian); transl: J. Soviet Math. **53**, no. 3 (1991), 252–263.

[7] L. V. Kovalev, *To the problem of extremal decomposition with free poles on the circle*, Far Eastern Math. Collection. **2** (1996), 96–98. (Russian)

[8] G. V. Kuzmina, *Extremal metric method in problems of the maximum of product of powers of conformal radii of non-overlapping domains with free parameters*, Notes scientific. sem. Leningr. Dep. of Math. Inst. AN USSR. **302** (2003), 52–67 (Russian); transl: Journal of Mathematical Sciences (New York) **129**, no. 3 (2005), 3843–3851.

[9] A. K. Bakhtin, G. P. Bakhtina, and Yu. B. Zelinskii, *Topological-algebraic structures and geometric methods in complex analysis*, Proceedings of the Institute of Mathematics of NAS of Ukraine **308** (2008). (Russian)

[10] Y. V. Zabolotnii, *The problem of calculating the maximum of product of inner radii of non-overlapping domains in the complex plane*, Proceedings of the Institute of Mathematics of NAS of Ukraine **10**, no. 4–5 (2013), 557–564. (Ukrainian)

[11] A. K. Bakhtin and I. V. Denega, *About one Dubinin's problem*, Proceedings of the Institute of Mathematics of NAS of Ukraine **10**, no. 4–5 (2013), 401–411. (Russian)

[12] A. K. Bakhtin and I. V. Denega, *Addendum to a theorem on extremal decomposition of the complex plane*, Bull. Soc. Sci. Lettr. Łódź, Sér. Recher. Déform. **62**, no. 2 (2012), 83–92.

[13] A. K. Bakhtin and I. V. Denega, *Separating transformation in problems of maximizing the product of powers of the inner radii of non-overlapping domains*, Proceedings of the Institute of Mathematics of NAS of Ukraine **9**, no. 2 (2012), 32–44. (Russian)

[14] J. A. Jenkins, *Univalent functions and conformal mapping*, Moscow: Publishing House of Foreign Literature **256** (1962). (Russian)

[15] P. M. Tamrazov, *Some extremal problems in the theory of univalent conformal mappings*, Math. Collection **67** (109), no. 3 (1965), 329–337. (Russian)

[16] P. M. Tamrazov, *Extremal conformal mappings and poles of quadratic differentials*, Mathematics of the USSR-Izvestiya **2**, no. 5 (1968), 987–996.

[17] N. A. Lebedev, *The Area Principle in the Theory of Univalent Functions*, Moscow: Science, 1975, 336 p. (Russian)

[18] G. P. Bakhtina, *Variational Methods and Quadratic Differentials in the Problems on Non-overlapping Domains*: summary of thesis ... candidate of physical and mathematical sciences: specialization 01.01.01 "Function Theory and functional analysis", Kyiv 1975, 11 p. (Russian)

[19] G. P. Bakhtina, *On extremalization some functionals in problems on non-overlapping domains*, Ukrainian Math. Journal **27**, no. 2 (1975), 202–204. (Russian)

[20] L. I. Kolbina, *Conformal mapping of the unit circle onto non-overlapping domains*, Bulletin of Leningrad University, **5** (1955), 37–43. (Russian)

[21] Y. V. Zabolotnii, *Application of separation transformation in one problem on non-overlapping domains*, Reports of NAS of Ukraine **9** (2011), 11–14. (Ukrainian)

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PRZEKSZTAŁCENIE ODDZIELAJĄCE I EKSPERYMENTALNY ROZKŁAD PŁASZCZYZNY ZESPOLONEJ

S t r e s z c z e n i e

Praca jest poświęcona zagadnieniom ekstremalnym geometrycznej teorii funkcji zmiennej zespolonej związanym z oszacowaniami funkcjonałów określonych na układach nie zachodzących na siebie obszarów. Badania są zogniskowane na znanym problemie Dubinina o ekstremalnym rozkładzie płaszczyzny zespolonej.

Słowa kluczowe: promień wewnętrzny, obszary nie zachodzące na siebie, “wolne” bieguny, układ n-radialny punktów, problem Dubinina, różniczka kwadratowa

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Janusz Garecki

CANONICAL SUPERENERGY DENSITY AND LOCAL GRAVITATIONAL STABILITY

Summary

In the framework of general relativity one has very hard problem with gravitational energy density, not satisfactorily solved up to now. It is a consequence of the Einstein Equivalence Principle. To avoid the problem we have introduced in past *the canonical superenergy tensors*. It turned out that these tensors give a very good tool to local, and in special cases also to global, analysis of the gravitational field in general relativity (See papers cited in References). Here we propose a new application of the canonical superenergy density. Namely, we propose to use this density to study local gravitational stability of the solutions to the Einstein equations. Our proposition follows the procedure of finding the stable minima of the interior energy U for a thermodynamical system. By using thermodynamical analogy we have formulated and proved Proposition from which it follows the Conclusion which says that when the total superenergy density, matter and gravitation, is *positive-definite*, then the solution can be gravitationally stable, i.e., it can be stable under small metric perturbations. Contrary, when the total superenergy density is *negative-definite*, then the solution cannot be gravitationally stable. We give many examples of application of the Conclusion.

Keywords and phrases: canonical superenrgy tensors, canonical superenergy density, gravitational stability

1. The canonical superenergy tensors

We begin with general remark that in the whole paper we use the same signature and notation as used in the last edition of the famous book by Landau and Lifshitz [7].

The Λ term which we consider in Section II we treat as source term in Einstein equations, i.e., as an energy-momentum tensor of the form

$$_{\Lambda}T_i^{\ k} = (-)\frac{\Lambda}{\beta}\delta_i^k, \quad \text{where} \quad \beta = \frac{8\pi G}{c^4}.$$

In the framework of general relativity (**GR**), as a consequence of the Einstein Equivalence Principle (**EEP**), the gravitational field *has non-tensorial strengths* $\Gamma^i_{kl} = \{^{\ i}_{kl}\}$ and *admits no energy-momentum tensor.* One can only attribute to this field *gravitational energy-momentum pseudotensors.* The leading object of such a kind is the *canonical gravitational energy-momentum pseodotensor* $_Et_i^{\ k}$ proposed already in past by Einstein. This pseudotensor is a part of the *canonical energy-momentum complex* $_EK_i^{\ k}$ in **GR**.

The canonical complex $_EK_i^{\ k}$ can be easily obtained by rewiriting Einstein equations to the superpotential form

$$(1) \qquad _EK_i^{\ k} := \sqrt{|g|}\big(T_i^{\ k} +_E t_i^{\ k}\big) =_F U_i^{\ [kl]}{}_{,l}$$

where $T^{ik} = T^{ki}$ is the symmetric energy-momentum tensor for matter, $g = det[g_{ik}]$, and

$$\begin{aligned} _Et_i^{\ k} &= \frac{c^4}{16\pi G}\big\{\delta_i^k g^{ms}\big(\Gamma^l_{mr}\Gamma^r_{sl} - \Gamma^r_{ms}\Gamma^l_{rl}\big) \\ &+ g^{ms}_{\ \ ,i}\big[\Gamma^k_{ms} - \frac{1}{2}\big(\Gamma^k_{tp}g^{tp} - \Gamma^l_{tl}g^{kt}\big)g_{ms} \\ &- \frac{1}{2}\big(\delta^k_s\Gamma^l_{ml} + \delta^k_m\Gamma^l_{sl}\big)\big]\big\}; \end{aligned}$$

$$(2) \qquad _FU_i^{\ [kl]} = \frac{c^4}{16\pi G}g_{ia}\big(\sqrt{|g|}\big)^{(-1)}\big[(-g)\big(g^{ka}g^{lb} - g^{la}g^{kb}\big)\big]_{,b}.$$

$_Et_i^{\ k}$ are components of the canonical energy-momentum pseudotensor for gravitational field

$$\Gamma^i_{kl} = \{^{\ i}_{kl}\}, \quad \text{and} \quad _FU_i^{\ [kl]}$$

are von Freud superpotentials.

$$(3) \qquad _EK_i^{\ k} = \sqrt{|g|}\big(T_i^{\ k} +_E t_i^{\ k}\big)$$

are components of the *Einstein canonical energy-momentu complex, for matter and gravity*, in **GR**.

In consequence of (1) the complex $_EK_i^{\ k}$satisfies local conservation laws

$$(4) \qquad _EK_i^{\ k}{}_{,k} \equiv 0.$$

In very special cases one can obtain from these local conservation laws the reasonable integral conservation laws.

Despite that one can easily introduce in **GR** *the canonical (and others) superenergy tensor* for gravitational field. This was done in past in a series of our articles (See, e.g., [3] and references therein). It appeared that the idea of the superenergy tensors is universal: to any physical field having an energy-momentum tensor or pseudotensor one can attribute the coresponding superenergy tensor.

So, let us give a short reminder of the general, constructive definition of the superenergy tensor $S_a{}^b$ applicable to gravitational field and to any matter field. The definition uses *locally Minkowskian structure* of the spacetime in **GR** and, therefore, it fails in a spacetime with torsion, e.g., in Riemann-Cartan spacetime.

In normal Riemann coordinates **NRC(P)** we define (pointwiese and in coordinate-free way)

$$S_{(a)}{}^{(b)}(P) = S_a{}^b := (-)\lim_{\Omega\to P}\frac{\int\limits_\Omega\left[T_{(a)}{}^{(b)}(y) - T_{(a)}{}^{(b)}(P)\right]d\Omega}{1/2\int\limits_\Omega\sigma(P;y)d\Omega}, \tag{5}$$

where

$$\begin{aligned} T_{(a)}{}^{(b)}(y) &:= T_i{}^k(y)e^i{}_{(a)}(y)e_k{}^{(b)}(y),\\ T_{(a)}{}^{(b)}(P) &:= T_i{}^k(P)e^i{}_{(a)}(P)e_k{}^{(b)}(P) = T_a{}^b(P)\end{aligned}$$

are *physical or tetrad components* of the pseudotensor or tensor field which describes an energy-momentum distribution, and $\{y^i\}$ are normal coordinates. $e^i{}_{(a)}(y), e_k{}^{(b)}(y)$ mean an orthonormal tetrad $e^i{}_{(a)}(P) = \delta^i_a$ and its dual $e_k{}^{(a)}(P) = \delta^a_k$ paralelly propagated along geodesics through P (P is the origin of the **NRC(P)**). We have

$$e^i{}_{(a)}(y)e_i{}^{(b)}(y) = \delta^b_a. \tag{6}$$

For a sufficiently small 4-dimensional domain Ω which surrounds **P** we require

$$\int\limits_\Omega y^i d\Omega = 0, \quad \int\limits_\Omega y^i y^k d\Omega = \delta^{ik}M, \tag{7}$$

where

$$M = \int\limits_\Omega (y^0)^2 d\Omega = \int\limits_\Omega (y^1)^2 d\Omega = \int\limits_\Omega (y^2)^2 d\Omega = \int\limits_\Omega (y^3)^2 d\Omega, \tag{8}$$

is a common value of the moments of inertia of the domain Ω with respect to the subspaces $y^i = 0,\ (i = 0,1,2,3)$. We can take as Ω, e.g., a sufficiently small analytic ball centered at P:

$$(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 \le R^2, \tag{9}$$

which for an auxiliary positive-definite metric

$$h^{ik} := 2v^i v^k - g^{ik}, \tag{10}$$

can be written in the form

$$h_{ik}y^i y^k \le R^2. \tag{11}$$

A fiducial observer **O** is at rest at the beginning **P** of the used Riemann normal coordinates **NRC(P)** and its four-velocity is $v^i =* \delta^i_o$. $=*$ means that an equations is valid only in special coordinates.

We would like to note that we always will take $e^i{}_{(o)} = v^i =* \delta^i_o$.

$\sigma(P;y)$ denotes the two-point *world function* introduced in past by J. L. Synge [4]

$$\sigma(P;y) = *\frac{1}{2}\big(y^{o^2} - y^{1^2} - y^{2^2} - y^{3^2}\big). \tag{12}$$

The world function $\sigma(P;y)$ can be defined covariantly by the *eikonal-like equation* [4]

$$g^{ik}\sigma_{,i}\sigma_{,k} = 2\sigma, \quad \sigma_{,i} := \partial_i\sigma, \tag{13}$$

together with

$$\sigma(P;P) = 0, \quad \partial_i\sigma(P;P) = 0. \tag{14}$$

The ball Ω can also be given by the inequality

$$h^{ik}\sigma_{,i}\sigma_{,k} \le R^2. \tag{15}$$

Tetrad components and normal components are equal at $\mathbf{P}$, so, we will write the components of any quantity attached to $\mathbf{P}$ without tetrad brackets, e.g., we will write $S_a{}^b(P)$ instead of $S_{(a)}{}^{(b)}(P)$ and so on.

If $T_i{}^k(y)$ are the components of an energy-momentum tensor of matter, then we get from (5)

$$_mS_a{}^b(P;v^l) = \big(2\hat{v}^l\hat{v}^m - \hat{g}^{lm}\big)\nabla_l\nabla_m\hat{T}_a{}^b = \hat{h}^{lm}\nabla_l\nabla_m\hat{T}_a{}^b. \tag{16}$$

Hat over a quantity denotes its value at $\mathbf{P}$, and ∇ means covariant derivative.

Tensor $_mS_a{}^b(P;v^l)$ is *the canonical superenergy tensor for matter.*

For gravitational field, substitution of the canonical Einstein energy-momentum pseudotensor as $T_i{}^k$ in (5) gives

$$_gS_a{}^b(P;v^l) = \hat{h}^{lm}\hat{W}_a{}^b{}_{lm}, \tag{17}$$

where

$$\begin{aligned} W_a{}^b{}_{lm} &= \frac{2\alpha}{9}\big[B^b{}_{alm} + P^b{}_{alm} \\ &- \frac{1}{2}\delta^b_a R^{ijk}{}_m\big(R_{ijkl} + R_{ikjl}\big) + 2\delta^b_a\beta^2 E_{(l|g}E^g{}_{|m)} \\ &- 3\beta^2 E_{a(l|}E^b{}_{|m)} + 2\beta R^b{}_{(a|g|l)}E^g{}_m\big]. \end{aligned}$$

Here

$$\alpha = \frac{c^4}{16\pi G} = \frac{1}{2\beta},$$

and

$$E_i{}^k := T_i{}^k - \frac{1}{2}\delta^k_i T \tag{18}$$

is the modified energy-momentum tensor of matter. In terms of $E_i{}^k$ Einstein equations read $R_i{}^k = \beta E_i{}^k$. If we admit Λ term then we will have $R_i{}^k = \beta E_i{}^k + \Lambda\delta^k_i$.

On the other hand

$$B^b{}_{alm} := 2R^{bik}{}_{(l|}R_{aik|m)} - \frac{1}{2}\delta^b_a R^{ijk}{}_l R_{ijkm} \tag{19}$$

are the components of the *Bel-Robinson tensor* (**BRT**), while

$$P^b{}_{alm} := 2R^{bik}{}_{(l|}R_{aki|m)} - \frac{1}{2}\delta^b_a R^{jik}{}_l R_{jkim} \tag{20}$$

is the Bel-Robinson tensor with "transposed" indices (ik).

Tensor ${}_gS_a{}^b(P;v^l)$ is the *canonical superenergy tensor* for gravitational field $\left\{{}^{\,i}_{kl}\right\}$.

In vacuum ${}_gS_a{}^b(P;v^l)$ takes the simpler form

$$ {}_gS_a{}^b(P;v^l) = \frac{8\alpha}{9}\hat{h}^{lm}\big(\hat{C}^{bik}{}_{(l|}\hat{C}_{aik|m)} - \frac{1}{2}\delta^b_a\hat{C}^{i(kp)}{}_{(l|}\hat{C}_{ikp|m)}\big). \tag{21}$$

Here $C^a{}_{blm}$ denote components of the *Weyl tensor.*

Some remarks are in order:

1. In vacuum the quadratic form ${}_gS_a{}^b v^a v_b$, where $v^a v_a = 1$, is *positive-definite* giving the gravitational *superenergy density* ϵ_g for a fiducial observer **O**.

2. In general, the canonical superenergy tensors are uniquely determined only along the world line of the observer **O**. But in special cases, e.g., in Schwarzschild spacetime or in Friedman universes, when there exists a physically and geometrically distinguished four-velocity $v^i(x)$, one can introduce in an unique way the unambiguous fields ${}_gS_i{}^k(x;v^l)$ and ${}_mS_i{}^k(x;v^l)$.

3. We have proposed in our previous papers to use the tensor ${}_gS_i{}^k(P;v^l)$ as a substitute of the non-existing gravitational energy-momentum tensor.

4. It can be easily seen that the superenergy densities

$$\epsilon_g :=_g S_i{}^k v^i v_k, \quad \epsilon_m :=_m S_i{}^k v^i v_k$$

   for an observer **O** who has the four-velocity v^i correspond exactly to the *energy of acceleration* $\frac{1}{2}m\vec{a}\vec{a}$ which is fundamental in Appel's approach to classical mechanics [5].

In past we have used the canonical superenergy tensors ${}_gS_i{}^k$ and ${}_mS_i{}^k$ to local (and also, in some cases, to global) analysis of well-known solutions to the Einstein equations like Schwarzschild and Kerr solutions; Friedman and Goedel universes, and Kasner and Bianchi I, II universes. The obtained results were interesting (See [3]), e.g., the total superenergy S for exterior Schwarzschild spacetime is immediately conected with Hawking temperature T of a Schwarzschild black hole.

We have also studied the transformational rules for the canonical superenergy tensors under conformal rescalling of the metric $g_{ik}(x)$ [3,6].

The idea of the superenergy tensors can be extended on angular momentum also [3]. The obtained angular supermomentum tensors do not depend on a radius vector and gravitational angular supermomentum tensor depends only on *spinorial part* of the gravitational angular momentum pseudotensor. We have used in our investigation the Bergmann-Thomson expression on angular momentum in general relativity.

2. Gravitational stability of the solutions to the Einstein equations and canonical superenergy density

By *local gravitational stability* we mean stability of a background metric $\tilde{g}_{ik}(x)$ under small perturbations, see, e.g., [7, 8]

$$g_{ik}(x) = \tilde{g}_{ik}(x) + h_{ik}(x), \tag{22}$$

where $|h_{ik}(x)| \ll |\tilde{g}_{ik}(x)|$.

This kind of stability is different from *Lyapunov's stability* connected with well-posed Cauchy problem and it is important for structure formation on a given background.

Recently we have observed an exciting correlation between the total superenergy density, $\epsilon_s := \epsilon_m + \epsilon_g$, and gravitational stability of the solutions to the Einstein equations. Namely, we have noticed that when a solution is stable at point **P**, then $\epsilon_s(P) \geq 0$, and when the solution is unstable, then $\epsilon_s(P) < 0$.

The examples of the above mentioned correlation

1. Exterior Schwarzschild with $\Lambda = 0$ — stable — $\epsilon_s > 0$:
2. Einstein static universe — unstable — $\epsilon_s < 0$;
3. Kerr solution with $\Lambda = 0$ — stable — $\epsilon_s > 0$;
4. Standard Friedman universes with $\Lambda = 0$ — stable — $\epsilon_s > 0$;
5. Exterior Reissner-Nordstroem with $\Lambda = 0$ — stable — $\epsilon_s > 0$;
6. Minkowski spacetime — stable — $\epsilon_s = 0$.

In the above examples **P** is an arbitrary point of the corresponding spacetime.

One can easily see that the above mentioned correlation follows from the Proposition.

Proposition. *If the canonical total energy density $K_0{}^0(y)$ has stable minimum at P, i.e., if P is stability point of the analyzed solution, $\tilde{g}_{ik}(y)$, then $\epsilon_s(P) > 0$.*

Proof. ⋆ Our proof lies on the constructive definition (5) and on the following thermodynamical fact: a stable minimum of the interior energy $U = U(S, V, N)$ is given by

$$\delta U = 0,\ \delta^2 U > 0. \tag{23}$$

We will apply the analogical conditions to the total canonical energy density, matter and gravitation, ${}_EK_0{}^0\big(g; g^{ik}; g^{ik}{}_{,l}; , g^{ik}{}_{,lm}\big)$ in **NRC(P)**. We use **NRC(P)** in our proof but we write the results covariantly. In consequence the ϵ_s has the same value as in the coordinates (x^i) which we use in our calculations under condition that we use the same vector $v^i v_i = 1$ calculating ϵ_s.

Namely, we put in **NRC(P)**

(24) $$\delta_E {K_0}^0(P) = 0, \quad \delta_E^2 {K_0}^0(P) > 0$$

as conditions on stable minimum of the ${}_E{K_0}^0(y)$ at the point P.

Small metric perturbations (22) do not destroy such minimum like as small variations δS, δV, δN do not destroy a local, stable minimum of U, i.e., the local minimum defined by (24) is a stable point of the considered background solution $\tilde{g}_{ik}(y)$.

It is seen from (5), (8), (12) that the sign of the superenergy density

$$ {S_a}^b(P) v^a v_b = \star \ \ {S_0}^0(P) $$

is determined by the sign of the integral in nominator of (5) because (-) *denominator is always positive*. Therefore, if ${K_0}^0(y)$ has stable minimum at point P, ie., if

$$\delta_E {K_0}^0(P) = 0, \ \delta_E^2 {K_0}^0(P) > 0, \quad \text{then} \ \ {S_0}^0(P) =_g {S_0}^0(P) +_m {S_0}^0(P) = \epsilon_s(P) > 0$$

in the case because ${K_{(0)}}^{(0)}(y) - {K_0}^0 > 0$. $\star$

From the Proposition it follows important Conclusion that ${S_0}^0(P) > 0$ is *necessary condition* for gravitational stability in P (We write this covariantly as $\epsilon_s(P) = {S_i}^k(P) v^i v_k > 0$.) $P \in \Omega$ is a running point of Ω.

From the Conclusion one has immediately that if ${S_0}^0(P) < 0$ in the domain Ω (We write this covariantly as $\epsilon_s(P) = {S_i}^k(P) v^i v_k < 0$, $P \in \Omega$.), then the considered solution cannot be gravitationally stable in Ω.

The stable flat Minkowskian spacetime gives an example of a limiting case with ${S_0}^0 = \epsilon_s = 0$.

Some examples of the application of the above Conclusion

1. De Sitter spacetime — $\epsilon_s < 0 \Longrightarrow$ The solution cannot be gravitationally stable.

2. Anti-de Sitter universe — $\epsilon_s < 0 \Longrightarrow$ The solution cannot be gravitationally stable.

3. Bianchi I universe with $\Lambda = 0$ — $\epsilon_s > 0 \Longrightarrow$ This solution can be gravitationally stable.

4. Kasner universe with $\Lambda = 0$ — $\epsilon_s > 0 \Longrightarrow$ This solution also can be gravitationally stable.

   In the above examples **P** is an arbitrary point of the solution.

5. Expanding dust Friedman universes with $k = 0$, $\Lambda < 0$: $\epsilon_s > 0$ for small values of the cosmic time t, and $\epsilon_s < 0$ for big values of t. It means that the solution can be stable only for small values of the cosmic time t.

6. Oscillating Friedman dust universes with $k = 0$, $\Lambda > 0$: $\epsilon_s > 0$ for bigger values of the scale factor $R(t)$ $\big(for\ t \in (\pi/3, \frac{5}{3}\pi)\big)$, and $\epsilon_s < 0$ for smaller

values of $R(t)$ $\big(for\ t \in (0, \frac{\pi}{3}; \frac{5}{3}\pi, 2\pi)\big)$. Thus, these solutions can be stable only for bigger values of the scale factor $R(t)$;

7. Expanding dust Friedman universe with $k = (-)1,\ \Lambda < 0$: $\epsilon_s > 0$ for small values of the cosmic time t, and $\epsilon_s < 0$ for big values of t. So, this solution, likely as in the case 5., can be gravitationally stable only for small values of t;

8. Exterior SdS static universe with $\Lambda > 0$ and SadS static universe with $\Lambda < 0$: $\epsilon_s < 0$ for big values of the radial coordinate r, and $\epsilon_s > 0$ for small values of r. We conclude from this that these solutions can be stable only for small values of r.

9. The Bertotti-Kasner spacetime with positive cosmological constant Λ: $\epsilon_s > 0$. So, this solution can be gravitationally stable.

10. A cosmic string in the Schwarzschild spacetime: $\epsilon_s > 0$. This solution also can be gravitationally stable.

11. Recently we are analyzing the Janis-Newman-Winicour spacetime which in spherical coordinates $(t, r, \vartheta, \varphi)$ reads

    $$ds^2 = a^{\gamma} c^2 dt^2 - a^{-\gamma} dr^2 - r^2 a^{1-\gamma}\big(d\vartheta^2 + \sin^2\vartheta d\varphi^2\big), \tag{25}$$

    where
    $$a := 1 - \frac{r_s}{\gamma \cdot r},\ r_s = \frac{2GM}{c^2},$$
    and the mass parameter M $r > r_H = \frac{r_s}{\gamma}$.

    For $\gamma = 1$ we obtain the standard Schwarzschild spacetime.

    As a preliminary result we have obtained that $\epsilon_s > 0$ for $0 < \gamma \leq 1$. Therefore, for $0 < \gamma \leq 1$ this solution can be gravitationally stable.

Concerning more detailed information about calculation of the superenergy densities cited above – see Apendix.

The our results concerning de Sitter and anti-de Sitter universes seem to be supported by the recent papers [1, 2, 9].

It is very interesting that following our Conclusion the gravitational stability of the considered dust Friedman models with $\Lambda \neq 0$ depends on the evolutional phase of these universes. It is sensible because $\Lambda < 0$ gives here a repulsive force which is growing with t and, therefore, should produce gravitational instability, and $\Lambda > 0$ gives an additional attractive force growing with $R(t)$ and thus strenthening gravitational stability.

3. Final remarks

On the *superenergy level* we have no problem with suitable tensor for gravity, e.g., one can introduce gravitational *canonical superenergy tensor*. The canonical superenergy tensors, gravitation and matter, are useful to local analysis of the solutions to the Einstein equations, especially to analyze of their singularities [3].

In this paper we have proposed a new application of the superenergy density to study local gravitational stability of the solution to the Einstein equations. As it was already mentioned, this kind of stability is different from Lyapunov's stability, which is connected with well-posed Cauchy problem [13](See also [14]), and it is important for structure formation on a given background. Our criterion of this stability has thermodynamical origin.

We think that the new application of the superenergy tensors can be useful.

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4. Appendix

We give here the canonical superenergy densities ϵ_s for de Sitter, anti-de Sitter, static Einstein and Reissner-Nordstöm universes, for some dust Friedman universes with cosmological constant Λ, for static SdS and SadS universes, for Bertotti-Kasner spacetime, for cosmic string in Schwarzschild spacetime, and for Janis-Newman-Winicour spacetime. For simplicity we will use in here the *geometrized units* in which $G = c = 1$.

As it was already mentioned we use the same notation and definitions as in [7], especially, the same form of the Einstein equations without or with cosmological term, and the same form of the FLRW line element.

The Λ term we always treat as source term in Einstein equations, i.e., as the energy-momentum tensor of the form

$$_{\Lambda}T_i^{\ k} = (-)\frac{\Lambda}{\beta}\delta_i^k.$$

1. De Sitter spacetime —

$$\epsilon_s = (-)\frac{28}{27}\alpha\Lambda^2 < 0.$$

Calculating ϵ_s we have used the the Lemaitre-Robertson form (in Cartesian coordinates) of the line element for de Sitter spacetime

$$ds^2 = dt^2 - e^{2kt}\left(dx^2 + dy^2 + dz^2\right), \tag{26}$$

where $k^2 = \frac{\Lambda}{3}$.

2. Anti-de Sitter spacetime —

$$\epsilon_s = (-)\frac{32}{27}\alpha\Lambda^2 < 0.$$

In this case we have used the line element in standard, static form

$$ds^2 = a^2\cosh^2 r dt^2 - a^2 dr^2 - a^2\sinh^2 r d\theta^2 - a^2\sinh^2 r\sin^2\theta d\varphi^2, \tag{27}$$

where $a = const > 0$, $\Lambda = (-)\frac{3}{a^2} < 0$.

3. Einstein static universe —

$$\epsilon_s = (-)\frac{4\alpha}{3R^4} < 0, \quad \text{where} \quad \frac{1}{R^2} = 4\pi(\rho + p) = \Lambda - 8\pi p > 0;$$

4. Exterior Reissner-Nordström spacetime —

$$\begin{aligned}\epsilon_s &= \frac{2\alpha}{9r^8}\left[3\left(2Q^2 - r_s r\right)^2 + 5\left(Q^2 - r_s r\right)^2 + 2\left(3Q^2 - r_s r\right)^2\right.\\ &+ \left.2\left(3Q^2 - r_s r\right)\left(2Q^2 - r_s r\right)\right]\\ &+ \frac{2Q^2}{r^8}\left(r_s r - 2Q^2\right) + \frac{12Q^2\Lambda_{RN}}{r^6}.\end{aligned}\tag{28}$$

The last expression is positive for

$$r \geq r_H = m + \sqrt{m^2 - Q^2},$$

i.e., outside and on horizon H of the Reissner-Nordström black hole.

Here

$$r_s := 2m, \quad \Lambda_{RN} := 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \quad \text{and} \quad m^2 > Q^2, \ \alpha = \frac{1}{16\pi}, \ \beta = 8\pi.$$

Concerning the line elements for Einstein static universe and for exterior Reissner-Nordstroem universe see [11] and [12] respectively.

5. FLRW dust universes with $\Lambda \neq 0$, $k = 0$. In this case

$$\epsilon_s = \frac{32\alpha}{3}\frac{\ddot{R}^2}{R^2} + \frac{284}{3}\alpha\frac{\dot{R}^4}{R^4} - 124\alpha\frac{\ddot{R}\dot{R}^2}{R^3} + 12\alpha\frac{\dot{R}\dddot{R}}{R^2}. \tag{29}$$

For $\Lambda < 0$, $k = 0$ one has the solution of the suitable Friedman equation [10]

$$R^3 = \frac{3C}{2\Lambda}\left[ch\{t(-3\Lambda)^{1/3}\} - 1\right], \quad C = \frac{8}{3}\pi\rho R^3 = const, \tag{30}$$

from which it follows

$$R(t) = At^{2/3} \tag{31}$$

for small t, and

$$R(t) = Be^{bt}, \tag{32}$$

for big values of t.

Here A, B, b denote suitable, positive constants.

Substituting the asymptotic values of $R(t)$ given by (31) and (32) into (29), one gets

$$\epsilon_s = \frac{9248\alpha}{243t^4} > 0, \tag{33}$$

for small t, and

$$\epsilon_s = (-)\frac{20}{3}\alpha b^4 < 0, \tag{34}$$

for big values of t.

For $\Lambda > 0,\ k = 0$, one has the oscillatory solution to the Friedman equation [10]

$$R(t) = A\big(1 - cosbt\big)^{1/3}, \tag{35}$$

where

$$A = \left(\frac{3C}{2\Lambda}\right)^{1/3}, \quad b = (3\Lambda)^{1/3}, \quad bt \in [0, 2\pi]. \tag{36}$$

In this case the formula (29) gives

$$\begin{aligned} \epsilon_s &= \frac{32\alpha}{27}\frac{b^4\cos^2 bt}{(1-\cos bt)^2} - \frac{1148\alpha b^4}{81}\frac{\cos bt \sin^2 bt}{(1-\cos bt)^3} \\ &+ \frac{1232\alpha b^4}{243}\frac{\sin^4 bt}{(1-\cos bt)^4} - \frac{4\alpha b^4}{3}\frac{\sin^2 bt}{(1-\cos bt)^2}. \end{aligned} \tag{37}$$

Again sign of the expression (37) depends on the evolutional phase of this universe: for bigger values of $R(t)$, i.e., for $t \in \left(\frac{\pi}{3}, \frac{5}{3}\pi\right)$, we have $\epsilon_s > 0$, and for smaller values of $R(t)$, i.e., for $t \in \left[(0, \frac{\pi}{3}) \cup (\frac{5}{3}\pi, 2\pi)\right]$ we have $\epsilon_s < 0$.

6. Friedman dust universe with $\Lambda < 0,\ k = (-)1$.

One gets in the case

$$\begin{aligned} \epsilon_s &= \frac{32\alpha}{3}\frac{\ddot{R}^2}{R^2} - \frac{4\alpha}{3R^2} - \frac{280\alpha}{3}\frac{\dot{R}^2}{R^4} \\ &+ \frac{284\alpha}{3}\frac{\dot{R}^4}{R^4} - 124\alpha\frac{\dot{R}^2\ddot{R}}{R^3} + 12\alpha\frac{\dot{R}\dddot{R}}{R^2} + 4\alpha\frac{\ddot{R}}{R^3}. \end{aligned} \tag{38}$$

Following Bondi [10] here we have

$$R(t) = At^{2/3} \tag{39}$$

for small values of t, and

$$R(t) = Be^{Dt}, \tag{40}$$

for big values of the cosmic time t.

Here $A,\ B,\ D$ mean suitable, positive constants.

Substituting the asymptotic values (39), (40) of the scale factor $R(t)$ into (38) one gets that

$$\epsilon_s = \frac{\alpha}{27t^4}1,688(8) > 0, \tag{41}$$

for small values of t, and

$$\epsilon_s = (-)\frac{29\alpha}{3}D^4 < 0, \tag{42}$$

for big values of the cosmic time t.

7. Static SdS universe with $\Lambda > 0$ and static SadS universe with $\Lambda < 0$.

   In this case

$$\begin{aligned}\epsilon_s &= \frac{12\alpha}{9r^4}\left[\frac{8}{3}\left(\frac{r_s}{2r} - \frac{\Lambda r^2}{3}\right)^2 + \frac{1}{4}\left(rr_s + \frac{\Lambda r^4}{3}\right)^2\right] \\ &+ \frac{4\alpha}{9}\left(\frac{r_s}{r^3} + \frac{\Lambda}{3}\right)^2 - \frac{4}{3}\alpha\Lambda^2,\end{aligned} \tag{43}$$

   where $r_s = 2m$.

   It is easily seen from (43) that for big values of the radial coordinate r (We leave only the terms with Λ in the case)

$$\epsilon_s = (-)\frac{20\alpha}{27}\Lambda^2 < 0, \tag{44}$$

   and for small values of r (We omit here the terms with Λ)

$$\epsilon_s = \frac{8\alpha}{3}\frac{r_s}{r^6} > 0. \tag{45}$$

8. Bertotti-Kasner spacetime [12] with cosmological constant $\Lambda > 0$ —

$$\epsilon_s = \frac{4\alpha}{9}\Lambda^2 > 0, \quad \text{where} \quad \alpha = \frac{1}{16\pi}.$$

9. A cosmic string in the Schwarzschild spacetime [12] —

$$\epsilon_s = \frac{8\alpha}{3}\frac{4M^2}{r^6} > 0.$$

   M is the mass of the black hole.

10. Janis-Newman-Winicour spacetime [12]. One gets in the case

$$\begin{aligned}\epsilon_s &= \frac{4\alpha}{9}\left[\frac{6r_s^2\left[2\gamma r - r_s(\gamma+1)\right]^2 a^{2\gamma-4}}{16\gamma^2 r^8} - \frac{r_s^4\left(1-\gamma^2\right)^2 a^{2\gamma-4}}{4\gamma^2 r^8}\right] \\ &+ \frac{2\alpha}{9}\left[\frac{12r_s^2\left[2\gamma^2 r - r_s(\gamma+1)\right]^2 a^{2\gamma-4}}{16\gamma^4 r^8} + \frac{6r_s^2\left[4\gamma^2 r - r_s\left(\gamma+1\right)^2\right]^2 a^{2\gamma-4}}{16\gamma^4 r^8}\right. \\ &+ \left.\frac{r_s^3\left[2\gamma r - r_s(\gamma+1)\right]\left(1-\gamma^2\right)a^{2\gamma-4}}{4\gamma^3 r^8}\right] + \frac{1}{\kappa}\frac{r_s^4\left(1-\gamma^2\right)\left(1+2\gamma a\right)a^{3\gamma-4}}{4\gamma^3 r^8}.\end{aligned} \tag{46}$$

    $\kappa = \frac{1}{2\alpha}$. Our preliminary result is that $\epsilon_s > 0$ for $0 < \gamma \leq 1$.

The total superenergy densities for the other solutions to the Einstein equations mentioned in this paper have been already given in past [3].

References

[1] D. Garfinkle, *AdS instability*, Matter of Gravity **39**, Winter (2012), 7.

[2] V. Emelyanov et al., Phys. Rev. D**86** (2012), 027302.

[3] J. Garecki, Rep. Math. Phys. **33** (1993), 57; **40** (1997), 485; **44** (1999) 95; Int. J. of Theor. Phys. **34** (1995), 2259; J. Math. Phys. **40** (1999), 4035; Ann. der Phys. (Leipzig), **11** (2002), 441; Ann. der Phys. (Berlin), **19** (2010), 263; M. P. Dąbrowski and J. Garecki, Class. Quantum Grav. **19** (2002), 1; J. Garecki, Class. Quantum Grav., **22** (2005), 4051; Found. of Physics **37** (2007), 341; M. P. Dąbrowski and J. Garecki, Phys. Letters B**686** (2010), 6; J. Garecki, in: *Gravitation–100 Years after Einstein in Prague*, eds. J. Bičak and T. Ledvinka, Springer Proceedings in Physics 157, Springer International Publishing Switzerland 2014.

[4] J. L. Synge, *Relativity: the General Theory*, North-Holland, Amsterdam 1960.

[5] P. Appel, Journal für die reine und angewandte Mathematik **121** (1900), 310; **122** (1900), 205; G. Białkowski, *Classical Mechanics*, PWN, Warsaw 1975 (in Polish).

[6] M. P. Dąbrowski, J. Garecki, and D. B. Blaschke, Ann. der Physik (Berlin) **18** (2009), 13.

[7] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields*, PWN, Warsaw 2009 (in Polish). There exists English translation.

[8] M. Demiański, *Relativistic Astrophysics*, PWN, Warsaw 1990 (in Polish). There exists English translation.

[9] P. Bizoń, *Is AdS stable?*, arXiv:1312.5544 [gr-qc].

[10] H. Bondi, *Cosmology*, second edition, Cambridge University Press, Cambridge 1961.

[11] R. C. Tolman, *Relativity, Thermodynamics and Cosmology*, Oxford at the Clarendon Press, Oxford 1966.

[12] T. Mueller and F. Grave, *Cataloque of Spacetimes*, arXiv: 0904.4184 [gr-qc].

[13] A. D. Rendall, *Partial Differential Equations in General Relativity*, Oxford Graduate Texts in Mathematics, Oxford University Press, Oxford 2008, Ch. 10.

[14] Ch. G. Boehmer and N. Chan, *Dynamical systems in cosmology*, arXiv:1409.5582[gr-qc].

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GĘSTOŚĆ KANONICZNEJ SUPERENERGII A LOKALNA STABILNOŚĆ GRAWITACYJNA

S t r e s z c z e n i e

W tej pracy pokazano, że gestość kanonicznej superenergii determinuje lokalną stabilność grawitacyjną rozwiązań równań Einsteina. Pokazano mianowicie, że rozwiązanie z dodatnią gęstością może być stabilne a rozwiązanie z ujemną gęstością kanonicznej superenergii jest lokalnie grawitacyjnie niestabilne. Rezultat ten może być ważny dla badań nad powstawaniem lokalnych struktur w danej czasoprzestrzeni, tj., na tle danego rozwiązania równań Einsteina.

Słowa kluczowe: tensory superenergii kanonicznej, gęstość superenergii kanonicznej, stabilność grawitacyjna

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to Professors J. Ławrynowicz and L. Wojtczak

Anatoly S. Serdyuk and Tetiana A. Stepanyuk

ESTIMATES FOR APPROXIMATIONS BY FOURIER SUMS, BEST APPROXIMATIONS AND BEST ORTHOGONAL TRIGONOMETRIC APPROXIMATIONS OF THE CLASSES OF (ψ, β)-DIFFERENTIABLE FUNCTIONS

Summary

We obtain the exact-order estimates for approximations by Fourier sums, best approximations and best orthogonal trigonometric approximations in metrics of spaces L_s, $1 \le s < \infty$, of classes of 2π-periodic functions, whose (ψ, β)-derivatives belong to unit ball of the space L_∞.

Keywords and phrases: Fourier sums, best approximations, best orthogonal trigonometric approximations

We denote by L_p, $1 \le p < \infty$, the space of 2π–periodic functions $f : \mathbb{R} \to \mathbb{C}$, summable to the power p on $[0, 2\pi)$, in which the norm is given by the formula

$$\|f\|_p = \Big(\int\limits_0^{2\pi} |f(t)|^p dt \Big)^{\frac{1}{p}};$$

and we denote by L_∞ the space of 2π-periodic measurable and essentially bounded functions $f : \mathbb{R} \to \mathbb{C}$ with the norm $\|f\|_\infty = \operatorname*{ess\,sup}_t |f(t)|$;

Let $f : \mathbb{R} \to \mathbb{R}$ be the function from L_1, whose Fourier series has the form

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx},$$

where

$$\hat{f}(k)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}f(t)e^{-ikt}dt$$

are Fourier coefficients of the function f, $\psi(k)$ is an arbitrary fixed sequence of real numbers and β is a fixed real number. Then, if the series

$$\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{\hat{f}(k)}{\psi(|k|)}e^{i(kx+\frac{\beta\pi}{2}\mathrm{sign}k)}$$

is the Fourier series of some function φ from L_1, then this function is called the (ψ,β)–derivative of the function f and denoted by f^{ψ}_{β}. A set of functions f, whose (ψ,β)–derivatives exist is denoted by L^{ψ}_{β} (see [1]).

If $f\in L^{\psi}_{\beta}$ and, at the same time, $f^{\psi}_{\beta}\in\mathfrak{N}$, where $\mathfrak{N}\subseteq L_1$, then we say that the function f belongs to the class $L^{\psi}_{\beta}\mathfrak{N}$. By $B_{R,p}$ we denote the balls of the radius R of real–valued functions from L_p, i.e., the sets

$$B_{R,p}:=\{\varphi:\ \mathbb{R}\to\mathbb{R},\ ||\varphi||_p\le R\},\quad R>0,\quad 1\le p\le\infty.$$

In present paper as $\mathfrak{N}$ we take the unit balls $B_{1,p}$. Herewith, the functional classes $L^{\psi}_{\beta}B_{1,p}$ are denoted by $L^{\psi}_{\beta,p}$.

In the case $\psi(k)=k^{-r}$, $r>0$, the classes $L^{\psi}_{\beta,p}$ are well–known Weyl–Nagy classes $W^{r}_{\beta,p}$.

For functions f from classes $L^{\psi}_{\beta,p}$ we consider: L_s–norms of deviations of the functions f from their partial Fourier sums of order $n-1$, i.e., the quantities

$$\|\rho_n(f;\cdot)\|_s=\|f(\cdot)-S_{n-1}(f;\cdot)\|_s,\quad 1\le s\le\infty, \tag{1}$$

where

$$S_{n-1}(f;x)=\sum_{k=-n+1}^{n-1}\hat{f}(k)e^{ikx};$$

best orthogonal trigonometric approximations of the functions f in metric of space L_s, i.e., the quantities of the form

$$e^{\perp}_m(f)_s=\inf_{\gamma_m}\|f(\cdot)-S_{\gamma_m}(f;\cdot)\|_s,\quad 1\le s\le\infty, \tag{2}$$

where γ_m, $m\in\mathbb{N}$, is an arbitrary collection of m integer numbers, and

$$S_{\gamma_m}(f;x)=\sum_{k\in\gamma_m}\hat{f}(k)e^{ikx};$$

and best approximations of the functions f in space L_s, i.e., the quantities of the form

$$E_n(f)_s=\inf_{t_{n-1}\in\mathcal{T}_{2n-1}}\|f-t_{n-1}\|_s,\quad 1\le s\le\infty, \tag{3}$$

where $\mathcal{T}_{2n-1}$ is the subspace of all trigonometric polynomials t_{n-1} with real coefficients of degrees not greater than $n-1$.

We set

$$\mathcal{E}_n(L^{\psi}_{\beta,p})_s = \sup_{f\in L^{\psi}_{\beta,p}} \|\rho_n(f;\cdot)\|_s, \quad 1\le p,s\le\infty, \tag{4}$$

$$e^{\perp}_n(L^{\psi}_{\beta,p})_s = \sup_{f\in L^{\psi}_{\beta,p}} e^{\perp}_n(f)_s, \ 1\le p,s\le\infty, \tag{5}$$

$$E_n(L^{\psi}_{\beta,p})_s = \sup_{f\in L^{\psi}_{\beta,p}} E_n(f)_s, \quad 1\le p,s\le\infty. \tag{6}$$

The following inequalities follow from given above definitions (4)–(6)

$$E_n(L^{\psi}_{\beta,p})_s \le \mathcal{E}_n(L^{\psi}_{\beta,p})_s, \quad e^{\perp}_{2n-1}(L^{\psi}_{\beta,p})_s \le \mathcal{E}_n(L^{\psi}_{\beta,p})_s, \ 1\le p,s\le\infty. \tag{7}$$

In present paper we solve the problem about finding the exact order estimates of the quantities $\mathcal{E}_n(L^{\psi}_{\beta,\infty})_s$, $E_n(L^{\psi}_{\beta,\infty})_s$ and $e^{\perp}_n(L^{\psi}_{\beta,\infty})_s$ for $1\le s<\infty$, $\beta\in\mathbb{R}$.

For the Weyl–Nagy classes the exact order estimates of the quantities $\mathcal{E}_n(W^r_{\beta,p})_s$ and $E_n(W^r_{\beta,p})_s$ are known for all admissible values of parameters r, p, s and β, i.e., for

$$r>\max\left\{\frac{1}{p}-\frac{1}{s},0\right\}, \quad \beta\in\mathbb{R} \quad \text{and} \quad 1\le p,s\le\infty$$

(see, e.g., [2, p. 47–49]). What concerning the best orthogonal trigonometric approximations $e^{\perp}_n(W^r_{\beta,p})_s$, so order estimates are known for them (see [3]–[9]) for various (but not for all possible) values of the parameters r,p,s and β.

Order estimates of the quantities (4)–(6) under certain restrictions for the parameters r,p,s and β were established in the works [1], [10]–[20]. However, the case $p=\infty$, $1\le s\le\infty$ for some or another reasons hasn't been investigated yet.

We denote by P the set of positive, almost decreasing sequences $\psi(k)$, $k\ge1$, (we remind, that sequence $\psi(k)$ almost decreases, if there exists a positive constant M such that for arbitrary $k_1\le k_2$ the following inequality is satisfied $\psi(k_2)\le M\psi(k_1)$) such that

$$\sup_{m\in\mathbb{N}} \sum_{k=2^m}^{2^{m+1}} |\psi_n(k+1)-\psi_n(k)| \le K\psi(n),$$

where

$$\psi_n(k)=\begin{cases} 0, & k<n,\\ \psi(k), & k\ge n,\end{cases}$$

and K is the quantity uniformly bounded with respect to n.

Theorem 1. *Let $\psi\in P$, $1\le s<\infty$ and $\beta\in\mathbb{R}$. Then*

$$E_n(L^{\psi}_{\beta,\infty})_s \asymp \mathcal{E}_n(L^{\psi}_{\beta,\infty})_s \asymp \psi(n). \tag{8}$$

Here and in what follows, we write $A(n)\asymp B(n)$ for postive sequences $A(n)$ and $B(n)$ to denote that there are positive constants K_1 and K_2 such that

$$K_1B(n)\le A(n)\le K_2B(n), \quad n\in\mathbb{N}.$$

Proof. At first let's prove that the following inequality is true

$$\mathcal{E}_n(L^{\psi}_{\beta,\infty})_s \leq K^{(1)}\psi(n), \quad 1 \leq s < \infty. \tag{9}$$

In inequality (9) and henceforth by $K^{(i)}$, $i = 1, 2, ...$ we denote quantities uniformly bounded with respect to n.

If $f \in L^{\psi}_{\beta,\infty}$, then

$$\|f^{\psi}_{\beta}\|_s \leq (2\pi)^{\frac{1}{s}} \|f^{\psi}_{\beta}\|_{\infty} \leq (2\pi)^{\frac{1}{s}}, \tag{10}$$

and so, it is obviously that

$$L^{\psi}_{\beta,\infty} \subset L^{\psi}_{\beta} B_{(2\pi)^{\frac{1}{s}},s} \subset L^{\psi}_{\beta} L_s, \ 1 \leq s < \infty. \tag{11}$$

The following proposition follows from the theorem 6.7.1 in [1].

Proposition 1. *Let $1 < s < \infty$, $\psi \in P$, $f \in L^{\psi}_{\beta} L_s$ and $\beta \in \mathbb{R}$. Then for arbitrary $n \in \mathbb{N}$ there exists a positive constant K, which is uniformly bounded with respect to n and f and such that*

$$\|\rho_n(f; x)\|_s \leq K\psi(n) E_n(f^{\psi}_{\beta})_s. \tag{12}$$

Taking into account (10), (11) and in view of proposition 1, we obtain the following estimates

$$\mathcal{E}_n(L^{\psi}_{\beta,\infty})_s \leq \mathcal{E}_n\big(L^{\psi}_{\beta} B_{(2\pi)^{\frac{1}{s}},s}\big)_s \leq (2\pi)^{\frac{1}{s}} K\psi(n), \quad 1 < s < \infty. \tag{13}$$

Thus, the inequalities (9) are proved for $1 < s < \infty$.

Let's show the rightness of correlation (9) for $s = 1$. We use the following statement (see, e.g., [2, p. 8]).

Proposition 2. *Let $1 \leq q \leq p \leq \infty$. On this if $f \in L_p$, then $f \in L_q$ and*

$$\|f\|_q \leq (2\pi)^{\frac{1}{q}-\frac{1}{p}} \|f\|_p. \tag{14}$$

By using (14) for $q = 1$, $p = 2$ and inequality (13) for $s = 2$, we obtain

$$\begin{aligned} \mathcal{E}_n(L^{\psi}_{\beta,\infty})_1 &= \sup_{f \in L^{\psi}_{\beta,\infty}} \|f(\cdot) - S_{n-1}(f;\cdot)\|_1 \\ &\leq (2\pi)^{\frac{1}{2}} \sup_{f \in L^{\psi}_{\beta,\infty}} \|f(\cdot) - S_{n-1}(f;\cdot)\|_2 = (2\pi)^{\frac{1}{2}} \mathcal{E}_n(L^{\psi}_{\beta,\infty})_2 \leq K^{(1)}\psi(n). \end{aligned} \tag{15}$$

The rightness of the inequality (9) follows from (13) and (15).

To obtain the lower bound of the quantity $E_n(L^{\psi}_{\beta,\infty})_s$, we consider the following function

$$f_1(t) = f_1(\psi; n; t) = \psi(n) \cos nt.$$

It is obvious, that $f_1 \in L^{\psi}_{\beta,\infty}$ and $f_1 \perp t_{n-1}$ for arbitrary $t_{n-1} \in \mathcal{T}_{2n-1}$. Therefore

$$\int\limits_{-\pi}^{\pi}(f_1(t)-t_{n-1}(t))\cos ntdt=\int\limits_{-\pi}^{\pi}f_1(t)\cos ntdt=\pi\psi(n)\quad\forall t_{n-1}\in\mathcal{T}_{2n-1}. \tag{16}$$

On the other hand, taking into account the proposition 2 for $q=1$, $p=s$, we get

$$\begin{aligned}\int\limits_{-\pi}^{\pi}(f_1(t)-t_{n-1}(t))\cos ntdt &\le \|f_1-t_{n-1}\|_1\\ &\le (2\pi)^{1-\frac{1}{s}}\|f_1-t_{n-1}\|_s,\quad 1\le s\le\infty,\quad\forall t_{n-1}\in\mathcal{T}_{2n-1}.\end{aligned} \tag{17}$$

In view of (16)–(17) we arrive at the inequalities

$$E_n(L^{\psi}_{\beta,\infty})_s\ge E_n(f_1)_s=\inf_{t_{n-1}\in\mathcal{T}_{2n-1}}\|f_1-t_{n-1}\|_s\ge\frac{1}{2}\psi(n),\quad 1\le s\le\infty. \tag{18}$$

Theorem 1 is proved.

We denote by B the set of positive sequences $\psi(k)$, $k\in\mathbb{N}$, for each of which there exists a positive constant K such that

$$\frac{\psi(k)}{\psi(2k)}\le K,\quad k\in\mathbb{N}.$$

The sequences

$$\psi(k)=k^{-r},\quad r>0,\quad \psi(k)=\ln^{-\varepsilon}(k+1),\ \varepsilon>0,$$

etc. are representatives of the set B.

Theorem 2. *Let $\psi\in P\cap B$, $1\le s<\infty$ and $\beta\in\mathbb{R}$. Then*

$$e^{\perp}_{2n}(L^{\psi}_{\beta,\infty})_s\asymp e^{\perp}_{2n-1}(L^{\psi}_{\beta,\infty})_s\asymp\psi(n). \tag{19}$$

Proof. It follows, from the formulas (7) and (9), that under the conditions of the theorem 1, next inequalities are true

$$e^{\perp}_{2n}(L^{\psi}_{\beta,\infty})_s\le e^{\perp}_{2n-1}(L^{\psi}_{\beta,\infty})_s\le\mathcal{E}_n(L^{\psi}_{\beta,\infty})_s\le K^{(1)}\psi(n). \tag{20}$$

Now we determine a lower bound of the quantity $e^{\perp}_{2n}(L^{\psi}_{\beta,\infty})_s$. For this we use the well–known result of Rudin–Shapiro (see, e.g., lemma 6.32.1 in [21]).

Proposition 3. *There exists sequence of numbers $\{\varepsilon_k\}_{k=0}^{\infty}$, such that $\varepsilon_k=\pm1$ and*

$$\Big\|\sum_{k=0}^{m}\varepsilon_k e^{ikx}\Big\|_{\infty}\le 5\sqrt{m+1},\quad m=0,1,... \tag{21}$$

Taking into account proposition 3 for $m=2n-1$, we choose the sequence of numbers

$$\{\xi_k\}_{k=0}^{\infty},\qquad \xi_k=\pm1$$

such that

$$\Big\| \sum_{k=0}^{2n-1} \xi_k e^{ikx} \Big\|_\infty \le 5\sqrt{2n}. \tag{22}$$

We set

$$\psi(0) := \psi(1)$$

and consider the function

$$f_2(t) = f_2(\psi; n; t) := \frac{1}{10\sqrt{2n}+2} \sum_{k=-2n+1}^{2n-1} \xi_{|k|} \psi(|k|) e^{ikt}. \tag{23}$$

Since, according to definition of (ψ, β)–derivative and the inequality (22),

$$\|(f_2)^\psi_\beta\|_\infty = \frac{1}{10\sqrt{2n}+2} \Big\| \sum_{k=1}^{2n-1} \xi_k e^{i(kt+\frac{\beta\pi}{2})} + \sum_{k=1}^{2n-1} \xi_k e^{i(-kt-\frac{\beta\pi}{2})} \Big\|_\infty$$

$$\le \frac{1}{10\sqrt{2n}+2} \Big(\Big\| \sum_{k=1}^{2n-1} \xi_k e^{i(kt+\frac{\beta\pi}{2})} \Big\|_\infty + \Big\| \sum_{k=1}^{2n-1} \xi_k e^{i(-kt-\frac{\beta\pi}{2})} \Big\|_\infty \Big)$$

$$= \frac{1}{5\sqrt{2n}+1} \Big\| \sum_{k=1}^{2n-1} \xi_k e^{ikt} \Big\|_\infty \le 1,$$

so $f_2 \in L^\psi_{\beta,\infty}$.

We consider the quantity

$$I = \inf_{\gamma_{2n}} \Big| \int_{-\pi}^{\pi} (f_2(t) - S_{\gamma_{2n}}(f_2; t)) \sum_{k=-2n+1}^{2n-1} \xi_{|k|} e^{ikt} dt \Big|.$$

By virtue of Holder's inequality, proposition 2 and correlation (22) for

$$1 \le s < \infty, \quad \frac{1}{s} + \frac{1}{s'} = 1,$$

$$I \le \inf_{\gamma_{2n}} \|f_2(t) - S_{\gamma_{2n}}(f_2; t)\|_s \Big\| \sum_{k=-2n+1}^{2n-1} \xi_{|k|} e^{ikt} \Big\|_{s'}$$

$$= e^\perp_{2n}(f_2)_s \Big\| \sum_{k=-2n+1}^{2n-1} \xi_{|k|} e^{ikt} \Big\|_{s'} \le (2\pi)^{\frac{1}{s'}} e^\perp_{2n}(f_2)_s \Big\| \sum_{k=-2n+1}^{2n-1} \xi_{|k|} e^{ikt} \Big\|_\infty$$

$$\le 2\pi e^\perp_{2n}(f_2)_s \Big(\Big\| \sum_{k=0}^{2n-1} \xi_k e^{ikt} \Big\|_\infty + \Big\| \sum_{k=1}^{2n-1} \xi_k e^{-ikt} \Big\|_\infty \Big) \tag{24}$$

$$\le 2\pi e^\perp_{2n}(f_2)_s \Big(2 \Big\| \sum_{k=0}^{2n-1} \xi_k e^{ikt} \Big\|_\infty + 1 \Big) \le 2\pi (10\sqrt{2n}+1)\ e^\perp_{2n}(f_2)_s.$$

On the other hand, taking into account the orthogonality of trigonometric system $\{e^{ikt}\}$ and the fact that $\xi_k^2 = 1$, we obtain

$$I = \frac{1}{10\sqrt{2n}+2}\inf_{\gamma_{2n}}\left|\int_{-\pi}^{\pi}\sum_{\substack{|k|\le 2n-1,\\ k\notin\gamma_{2n}}}\xi_{|k|}\psi(|k|)e^{ikt}\sum_{k=-2n+1}^{2n-1}\xi_{|k|}e^{ikt}dt\right|$$

(25)

$$= \frac{\pi}{5\sqrt{2n}+1}\inf_{\gamma_{2n}}\sum_{\substack{|k|\le 2n-1,\\ k\notin\gamma_{2n}}}\psi(|k|).$$

Since the sequence $\psi(k)$ almost decreases, so

$$\inf_{\gamma_{2n}}\sum_{\substack{|k|\le 2n-1,\\ k\notin\gamma_{2n}}}\psi(|k|) \ge K^{(2)}\inf_{\gamma_{2n}}\sum_{\substack{|k|\le 2n-1,\\ k\notin\gamma_{2n}}}\psi(2n-1) = K^{(2)}\psi(2n-1)(2n-1). \tag{26}$$

In view of (24)–(26) we get

$$e^{\perp}_{2n}(f_2)_s \ge \frac{K^{(2)}\psi(2n-1)(2n-1)}{(10\sqrt{2n}+2)(10\sqrt{2n}+1)} \ge K^{(3)}\psi(2n). \tag{27}$$

Since, if $\psi\in B$, so $\psi(2n)\ge K^{(4)}\psi(n)$, and, hence, taking into account (27), we find

$$e^{\perp}_{2n}(L^{\psi}_{\beta,\infty})_s \ge e^{\perp}_{2n}(f_2)_s \ge K^{(5)}\psi(n). \tag{28}$$

Estimates (19) follow from (20) and (28). Theorem 2 is proved.

Corollary 1. *Let $r>0$, $1\le s<\infty$ and $\beta\in\mathbb{R}$. Then*

$$e^{\perp}_{2n}(W^r_{\beta,\infty})_s \asymp e^{\perp}_{2n-1}(W^r_{\beta,\infty})_s \asymp n^{-r}. \tag{29}$$

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References

[1] A. I. Stepanets, *Methods of Approximation Theory*, VSP: Leiden, Boston 2005.

[2] V. N. Temlyakov, *Approximation of Periodic Function*, Nova Science Publichers, Inc., (1993).

[3] E. S. Belinsky, *Approximation by a "floating" system of exponents on the classes of periodic functions with bounded mixed derivative*, Issled. po teorii func. mnog. vesch. perem., Jaroslavl': Jaroslav. un–t., (1988), 16–33. [in Russian]

[4] V. N. Temlyakov, *Approximations of functions with bounded mixed derivative*, Trudy Mat. Inst. Steklov. **178** (1986), 3–113 [in Russian]; Eng. transl.: Proceedings of the Steklov Institute of Mathematics **178** (1989), 1–121.

[5] B. S. Kashin and V. N. Temlyakov, *On best m–term approximations and the entropy of sets in the space L_1*, Mat. Zametki, **56**, no. 5 (1994), 57–86 [in Russian]; Eng. transl.: Mathematical Notes, **56**, no. 5 (1994), 1137–1157.

[6] A. S. Romanyuk, *Approximation of the classes $B^r_{p,\Theta}$ of periodic functions of several variables by partial Fourier sums with arbitrary choice of harmonics*, Proc. of the Institute of Mathematics "Fourier series: theory and applications", Ukrainian National Academy of Sciences [in Russian], Kyiv 1992, 112–118.

[7] A. S. Romanyuk, *Approximation of classes of periodic functions in several variables*, Mat. Zametki **71**, no. 1 (2002), 109–121 [in Russian]; Eng. transl.: Mathematical Notes **71**, no. 1 (2002), 98–109

[8] A. S. Romanyuk, *Best trigonometric approximations for some classes of periodic functions of several variables in the uniform metric*, Mat. Zametki **82**, np. 2 (2007), 247–261 [in Russian]; Eng. transl.: Mathematical Notes **82**, no. 2 (2007), 216–228.

[9] A. S. Romanyuk, *Approximation characteristics of classes of periodic functions of several variables*, Proc. of the Institute of Mathematics, Ukrainian National Academy of Sciences [in Russian], Kyiv, **93** (2012). [in Russian]

[10] A. I. Stepanets and A. K. Kushpel, *Convergent rate of Fourier series and best approximations in the space L_p*, Ukr. Mat. Zh., **39**, no. 4 (1987), 483–492 [in Russian]; Eng. transl.: Ukr. Math. J. **39**, no. 4 (1987), 389–398.

[11] U. Z. Hrabova and A. S. Serdyuk, *Order estimates for the best approximations and approximations by Fourier sums of the classes of (ψ, β)–differential functions*, Ukr. Mat. Zh. **65**, no. 9 (2013), 1186–1197 [in Ukrainian]; Eng. transl.: Ukr. Math. J. **65**, no. 9 (2013), 1319–1331.

[12] A. S. Serdyuk and T. A. Stepanyuk, *Order estimates for the best approximations and approximations of the classes of infinitely differential functions by Fourier sums*, Proc. of the Institute of Mathematics, Ukrainian National Academy of Sciences [in Ukrainian], Kyiv, **10**, no. 1 (2013), 255–282.

[13] A. S. Serdyuk and T. A. Stepanyuk, *Estimates for the best approximations of the classes of infinitely differentiable functions in uniform and integral metrics*, Ukr. Mat. Zh., **66**, no. 9 (2014), 1244–1256 [in Ukrainian]; Eng. transl.: Ukr. Math. J., **66**, no. 9 (2015), 1393–1407.

[14] T. A. Stepanyuk, *Estimates for the best approximations and approximations by Fourier sums of the classes of convolutions of periodic functions of not high smoothness in the integral metrics*, Proc. of the Institute of Mathematics, Ukrainian National Academy of Sciences [in Ukrainian], Kyiv **11**, no. 3 (2014), 241–269.

[15] A. S. Serdyuk and T. A. Stepanyuk, *Order estimates for the best approximations and approximations by Fourier sums in the classes of convolutions of periodic functions of low smoothness in the uniform metric*, Ukr. Mat. Zh., **66**, no. 12 (2014), 1658–1675 [in Ukrainian]; Eng. transl.: Ukr. Math. J. **66**, no. 12 (2015), 1862–1882.

[16] O. S. Fedorenko, *On the best m–term trigonometric and orthogonal trigonometric approximations*, Ukr. Mat. Zh. **51**, no. 12 (1999), 1719–1721 [in Ukrainian]; Eng. transl.: Ukr. Math. J. **51**, no. 12 (2000), 1945–1949.

[17] O. S. Fedorenko, *Approximation of (ψ, β)–differentiable functions by trigonometric polynomials*: Autoref. dissertation ... cand. phys.–math. sciences, Kyiv: Insitute of Mathematics of NAS of Ukraine (2001). [in Ukrainian]

[18] V. V. Shkapa, *Best orthogonal trigonometric approximations of functions of classes $L^{\psi}_{\beta,1}$*, Proc. of the Institute of Mathematics, Ukrainian National Academy of Sciences [in Ukrainian], Kyiv, **11**, no. 3 (2014), 315–329.

[19] V. V. Shkapa, *Estimates of the best M–term and orthogonal trigonometric approximations of functions of classes $L^{\psi}_{\beta,p}$ in the uniform metrics*, Proc. of the Institute of Mathematics, Ukrainian National Academy of Sciences [in Ukrainian], Kyiv, **11**, no. 2 (2014), 305–307.

[20] A. S. Serdyuk and T. A. Stepanyuk, *Order estimates for the best orthogonal trigonometric approximations of the classes of convolutions of periodic functions of low smoothness in the uniform metric*, Ukr. Mat. Zh. **66**, no. 12 (2014), 1658–1675 [in Ukrainian].

[21] M. I. Dyachenko and P. L. Ulyanov, *Measure and integral*, "Factorial", Moscow 1998. [in Russian]

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Presented by Adam Paszkiewicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 16, 2015

OSZACOWANIA DLA APROKSYMACJI SUMAMI FOURIERA, NAJLEPSZYCH APROKSYMACJI I NAJLEPSZYCH ORTOGONALNYCH TRYGONOMETRYCZNYCH APROKSYMACJI KLAS FUNKCJI RÓŻNICZKOWALNYCH

S t r e s z c z e n i e

Uzyskujemy oszacowania dokładne co do rzędu aproksymacji sumami Fouriera i najlepszych ortogonalnych trygonometrycznych aprksymacji w metrykach przestrzeni L_s, $1 \leq s < \infty$, klas funkcji 2π-okresowych, których (ψ, β)-pochodne należą do kuli jednostkowej w przestrzeni L_∞.

Słowa kluczowe: sumy Fouriera, najlepsze aproksymacje, najlepsze ortogonalne trygonometryczne aproksymacje

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Contribution to the jubilee volume, dedicated to Professors J. Ławrynowicz and L. Wojtczak

Osamu Suzuki

BINARY AND TERNARY STRUCTURES IN PHYSICS I

THE HIERARCHY STRUCTURE OF TURING MACHINE IN PHYSICS

Summary

We recall some basic facts on Turing machine and observe its hierarchy structure. Associating the particle and anti-particle to the push down automaton and three quarks to linear bounded automaton, respectively. We can describe quark physics in terms of Turing machine.

Keywords and phrases: binary structure, ternary structure, Turing machine vs. particle systems, bosonic field, push down automaton type, ternary and cubic algebras

I. Binary and ternary structures in physics

In this section we give several binary and ternary structures in physics. We shall treat the following topics:

(1) Quark physics.

(2) Atomic physics.

(3) The basic problem on the space-time.

1. Quark physics

In the theory of elementary particles we can observe several binary and ternary phenomena.

(4) Particle and anti-particle.

(5) 3-generation in quarks.

(6) Mesons and baryons.

In Part II we shall be concerned with these structures in detail.

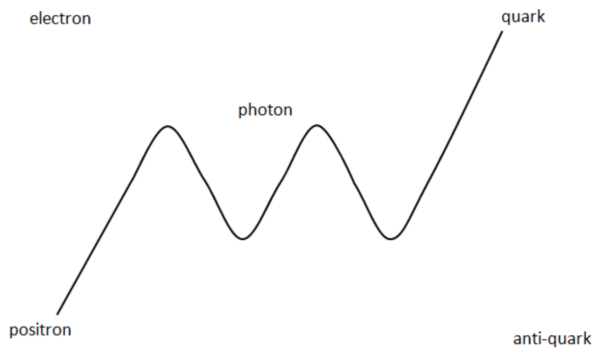

Fig. 1: Photon in quark physics.

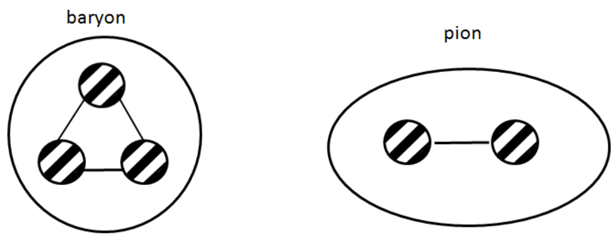

Fig. 2: Ternary and binary structures of baryons and pions.

2. Atom physics

We know that atoms and molecules constitute with hydrogen H helium He and other atoms. We shall find binary and ternary structures in generations of molecules.

The common understanding on the generations of atoms can be given as follows:

2.1. Binary generation

(7) Proton P and neutron n are created as stable particles.

(8) A combination of a proton and a neutron happens and deuterium is created.

(9) Another combination of a deuterium and a proton is created and tritium is obtained.

(10) A combination of a proton and tri-heavy hydrogen happens and a helicum H_4 is created.

(11) He is stable and it remains in the universe.

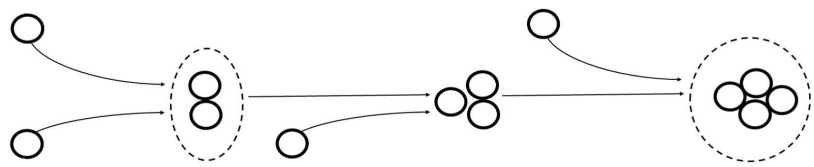

Fig. 3: From deuterium to helium.

2.2. Ternary generation

In the star another generation scheme starts and carbon ^{12}C arises from helium ^{4}He by way of the construction of berylium ^{8}Be:

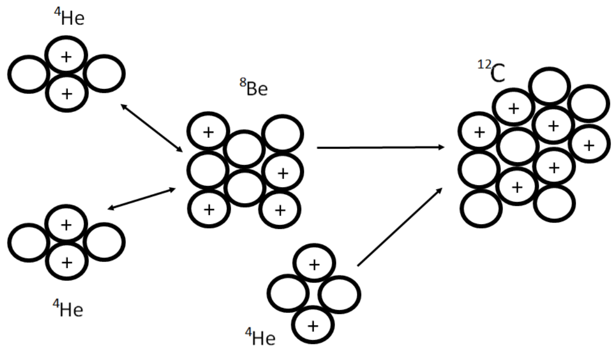

Fig. 4: From helium to beryllium.

By these we have the following generations shown in Fig. 5 and Fig. 6.

2.3. Generation of the space-time

By this method we may expect to treat the fundamental problem of the space-time:

(12) How can we describe the birth of time? Haw can we describe the past and the future?

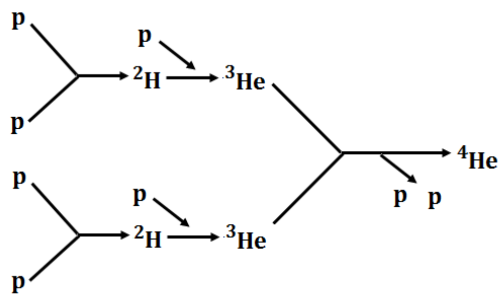

Fig. 5: Proton-proton chain.

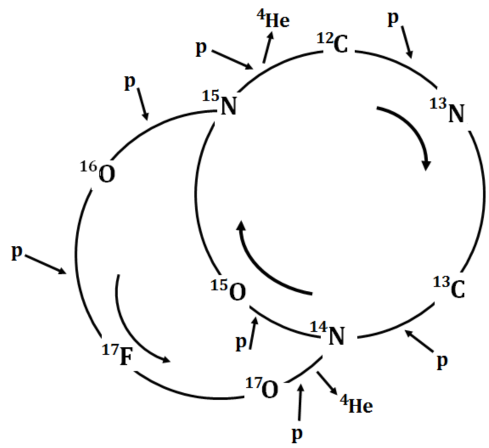

Fig. 6: CNO cycle.

(13) How can we describe the birth of the space? How can we understand that the space has three dimensions?

Penrose, Hawking and others have discussed these problems. We treat these problems in terms of the Turing machine and compare the results with their results [1, 2].

The daily experiences tell that the time has dimension 1 and the space has dimension 3. We may discuss the space-time by Turing machine. We may imagine the following process in the generation. This observation is similar to that done by Penrose [1].

(14) At first the discrete time is born by the "beat of vacuum" with the frequencey $E = h\nu$. The photons are created by this and the universe was born.

(15) Then the conjugation of the time created which might be connected to the past. This was created by a push-down automaton.

The future and the past can be made different in terms of the entropy. The big-bang can be understood as the increase of the entropy. The past can be described in terms of the decrease of the entropy. Hence we have a different conjugation in the theory of special relativity:

$$t \longleftrightarrow -t.$$

The time in this discussion is similar to that the philosophy of E. Bergson [1].

(16) The creation of the space can be performed by the time. The distance can be measured by the counting of the beats. When the time is filled, i.e. the time is periodic and the period is determined.

Then the lower bound automaton starts and the dimensions are created. The basic creation tells the dimension 3:

$$(x, y, z)\,.$$

(17) The mixture of the space and time happens and the special relativity can be imagined.

II. The hierarchy structure in Turing machine

In this section we recall a hierarchy structure in Turing machine and find the back around of the binary and ternary structure.

Hierachy structure

There exist 4 kind of the automatons in Turing machine, which are called types:

(18) Finite automaton ($= \mathcal{L}_3$).

(19) Push-down automaton ($= \mathcal{L}_2$).

(20) Linear bounded automaton ($= \mathcal{L}_1$).

(21) General Turing machine ($= \mathcal{L}_0$).

Then we can see the following facts:

Proposition 1.

(22) $\mathcal{L}_3 \subsetneqq \mathcal{L}_2 \subsetneqq \mathcal{L}_1 \subsetneqq \mathcal{L}_0$.

(23) *A Turing machine belongs to one and only one type.*

m-Words system

We consider the following system which is called *m-words system*:

$$\mathcal{L}^{(m)} = \{a_1^n a_2^n \ldots a_m^n | n > 0\}\,.$$

We consider Turing machines which accept $\mathcal{L}^{(m)}$ as acceptable sentences. In this paper we identify the system with the Turing machine.

(I) $\mathcal{L}^{(1)}$

$\mathcal{L}^{(1)}$ is a finite automaton. Moreover, putting

$$\tilde{\mathcal{L}^{(k)}} = \mathcal{L}^{(1)'} \cdot \mathcal{L}^{(1)''} \cdot \ldots \cdot \mathcal{L}^{(1)^{(h)}} = \{a_1^{m_1} a_2^{m_2} \ldots a_k^{m_k} | m_1 \cdot \ldots \cdot m_k > 0\}$$

we have general finite automaton. The transition diagram of $\mathcal{L}^{(1)}$ is given in Fig. 7.

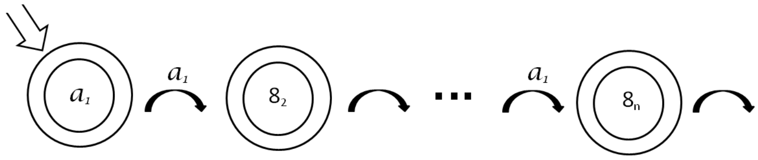

Fig. 7: The automaton $\mathcal{L}^{(1)}$.

(II) $\mathcal{L}^{(2)}$

$\mathcal{L}^{(2)}$ is a push-down automaton. Hence we can see that $\mathcal{L}^{(2)} \in \mathcal{L}_2$. The important fact is that

$$\mathcal{L}^{(2)} \notin \mathcal{L}_3 \quad \text{and} \quad \mathcal{L}^{(2)} \in \mathcal{L}_2.$$

We can give several realizations of the automatons:

(24) The context free sentence

$\langle N, \Omega, P, S \rangle$ where $N = \{S\}$, $\Omega = \{a, b\}$, $P = \{S \to aSb,\ S \to ab\}$.

Generation of sentence $a^2 b^2$

1. $S \to aSb$,

2. $S \to a(ab)b = a^2 b^2$.

(25) Push-down automaton

We give the transition diagram (Fig. 8). The detailed explanations are omitted.

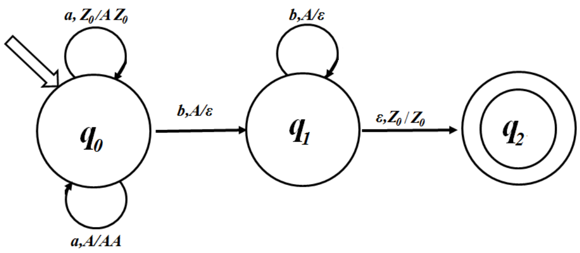

Fig. 8: The push-down automaton $\mathcal{L}^{(2)}$.

Transition of $aabb$.

$$
\begin{array}{cc}
(q_0, \mathcal{L}_0) & (q_1, A) \\
\Downarrow a & \Downarrow b \\
(q_b, A) & (q_1, \varepsilon) \\
\Downarrow a & \Downarrow \varepsilon \\
(q_0, AA) & (q_f, \mathcal{L}_0) \\
\Downarrow b &
\end{array}
$$

(26) Turing machine

We give the Turing machine description. The basic idea can be described by the following configuration (Fig. 9).

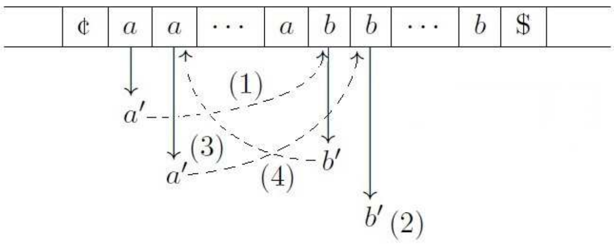

Fig. 9: The Turing machine automaton $\mathcal{L}^{(2)}$.

1. We start from the first a and marked a'.
2. We proceed to the right to find the b and marked b'.
3. We proceed to the left and find a except a' and mark a'.
4. We proceed to the right and find b (except b') and marked b'.

The transition diagram is given in Fig. 10.

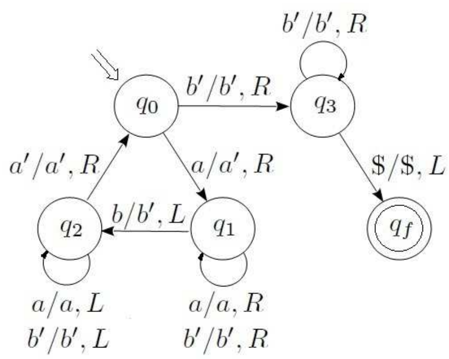

Fig. 10: The transition diagram for automaton $\mathcal{L}^{(2)}$.

Transition of $abab$.

$$
\begin{array}{cc}
(1) & (aabb : q_0) \\
& \Downarrow \\
(2) & (a'abb : q_0) \\
& \Downarrow R \\
(3) & (a'abb : q_1) \\
& \Downarrow \\
(4) & (a'ab'b : q_1) \\
& \Downarrow L \\
(5) & (a'ab'b : q_2) \\
& \Downarrow \\
(6) & (a'a'b'b' : q_0) \\
& \Downarrow \\
(7) & (a'a'b'b' : q_3) \\
& \Downarrow \\
(8) & (a'a'b'b' : q_f)
\end{array}
$$

(III) $\mathcal{L}^{(m)}$ $(m \geq 3)$

$$\mathcal{L}^{(m)} \in \mathcal{L}_1, \quad \mathcal{L}^{(m) \notin \mathcal{L}_2}.$$

The basic type of $\mathcal{L}^{(m)}(m \geq 3)$ is $\mathcal{L}^{(3)}$. We can show that $\mathcal{L}^{(m)}$ can be generated by $\mathcal{L}^{(3)}$. The Turing machine for $\{a^n b^n a^n | n > 0\}$ can be given as in Fig. 11.

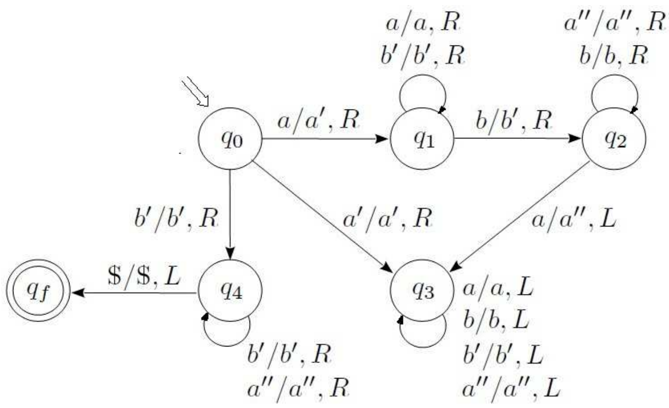

Fig. 11: The automaton $\mathcal{L}^{(m)}$, $(m \geq 3)$.

Transition of *aba*.

$$
\begin{array}{cc}
(1) & (abc : q_0) \\
 & \Downarrow R \\
(2) & (a'ba : q_1) \\
 & \Downarrow R \\
(3) & (a'b'a : q_2) \\
 & \Downarrow L \\
(4) & (a'b'a'' : q_3) \\
 & \Downarrow R \\
(5) & (a'b'a'' : q_0) \\
 & \Downarrow R \\
(6) & (a'b'a'' : q_4) \\
 & \Downarrow L \\
(7) & (a'b'a'' : q_f)
\end{array}
$$

Next we give the basic idea of the construction (Fig. 12).

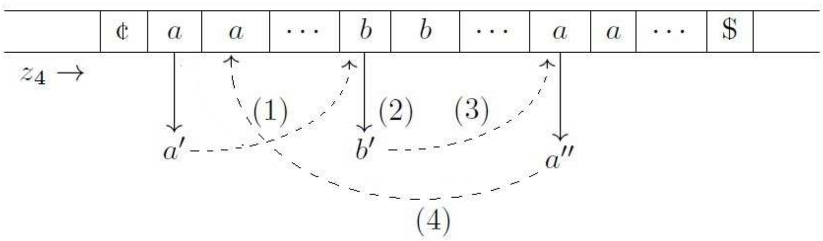

Fig. 12: The basic idea of construction of the automaton $\mathcal{L}^{(m)}$, $(m \geq 3)$.

From the action we can see that (1) The Turing machine is a lower bound automaton. (2) Replacing a in the third part with c, a'' with c'' respectively, we can obtain the Turing machine for $\mathcal{L}^{(3)}$ in a completely analogous manner. (3) We can see easily that the Turing machine for $\mathcal{L}^{(m)}$ $(m \geq 4)$ can be constructed in a completely analogous manner. Hence we can see that $\mathcal{L}^{(m)}$ is also a lower bound automaton.

(IV) General Turing machine

The Turing machine which does not belong to one of the above three types belongs to general Turing machine. We notice the Turing machine which describes mathematical operations "$a + b$". "ab" belongs to this type. This automaton is not lower bound automaton. In fact we have an "overflow" in the tape:

$$
\begin{array}{r}
|0|\ | \\
+)0|\ |\ | \\
- - -- \\
|00|\ 0
\end{array}
$$

III. Binary and ternary structures in Turing machine

In this section we introduce a concept of evolution in Turing machine and find the binary and ternary structures in it [3, 4].

Evolution in m-words system

Here we introduce the concept of evolution of $\mathcal{L}^{(m)}$ $(m = 1, 2, 3)$. We consider the evolution:

$$\mathcal{L}^{(m-1)} \longrightarrow \mathcal{L}^{(m)}.$$

The embedding is given by

$$\mathcal{L}^{(m-1)'} = \left\{a_1^n \dots a_{m-1}^n a_m^k | n > 0,\ k > 0\right\} = \mathcal{L}^{(m)} \cdot \mathcal{L}^{(1)},$$

where $X \cdot Y$ implies the connection. Here we notice the following proposition:

Proposition 2. *$\mathcal{L}^{(m)}$ and $\mathcal{L}^{(m)'}$ have the same type. Then we have the following sequence:*

$$\mathcal{L}^{(1)} \longrightarrow \mathcal{L}^{(2)} \longrightarrow \mathcal{L}^{(3)} \longrightarrow \dots.$$

We call the evolution *non-trivial* when the following holds:

$\mathcal{L}^{i-1)}$ and $\mathcal{L}(i)$ do not belong the same type of the Turing machine. Then we can prove the following:

Proposition 3.

(27) $\mathcal{L}^{(1)} \to \mathcal{L}^{(2)}$ is non-trivial.

(28) $X^{(2)} \to \mathcal{L}^{(3)}$ is non-trivial.

(29) $\mathcal{L}^{(M)} \to \mathcal{L}^{(m+1)}$ $(m \geq 3)$ is trivial.

Three generation of m-words system

Next we proceed to the generation of the system. Here we introduce a concept of generations of evolutions in the systems. Putting

$$\mathcal{L}^{(2)'} = \{a_1^m a_2^n a_3^n | m, n > 0\},$$
$$\mathcal{L}^{(2)'''} = \{a_1^n a_2^n a_3^m | n, m > 0\},$$

we can see that $\mathcal{L}^{(2)'} \in \mathcal{L}_2$, $\mathcal{L}^{(2)'''} \in \mathcal{L}_2$ and

$$\mathcal{L}^{(3)} = \{a_1^n a_2^n a_3^n | n > 0\},$$
$$\mathcal{L}^{(3)} = \mathcal{L}^{(2)'} \cap \mathcal{L}^{(2)'''}.$$

We call these phenomena *the generation* of $\mathcal{L}^{(3)}$ by $\mathcal{L}^{(2)'}$ and $\mathcal{L}^{(2)''}$. Next we consider the generation of $\mathcal{L}^{(4)}$

$$\mathcal{L}^{(4)} = \{a_1^n a_2^n a_3^n a_4^n | n > 0\}.$$

Then we have the following generation:

$$\dot{\mathcal{L}}^{(4)} = \mathcal{L}^{(3)'} \cap \mathcal{L}^{(3)''}$$

where

$$\mathcal{L}^{(3)'} = \{a_1^m a_2^n a_3^n a_4^n | m > 0, n > 0\}\,,$$
$$\mathcal{L}^{(3)''} = \{a_1^n a_2^n a_3^m a_4^m | m > 0, n > 0\}\,.$$

Here we want to make a strong stress on the following facts:

(30) $\mathcal{L}^{(3)} = \mathcal{L}^{(2)'} \cap \mathcal{L}^{(2)''}$ has a generation jump, i.e.,

$$\mathcal{L}^{(2)'}, \quad \mathcal{L}^{(2)''} \in \mathcal{L}_{(2)}, \quad \mathcal{L}^{(3)} \notin \mathcal{L}_{(2)}, \quad \mathcal{L}^{(3)} \in \mathcal{L},$$

(31) $\mathcal{L}^{(4)} = \mathcal{L}^{(3)'} \cap \mathcal{L}^{(3)''}$ has no generation jump and keeps the type of the Turing machine.

When the generation keeps its type we call it (*type-*)*preserving generation.* Then we can prove the following:

Proposition 4. *The generation $\mathcal{L}^{(m)} = \mathcal{L}^{(m-1)'} \cap \mathcal{L}^{(m-1)''}$ is non-preserving for $m = 3$ and preserving $m \geq 4$. Hence $\mathcal{L}^{(m)}(\geq 4)$ can be generated by $\mathcal{L}^{(3)}$ successively.*

On the basis of this proposition we can discuss the binary and ternary structure in the nature. Namely we can prove the following:

Theorem I.

(32) *Every Turing machine belongs to one and only one type: $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$.*

(33) *Two words system belongs to $\mathcal{L}_2$.*

(34) *Three-words system belongs to $\mathcal{L}_1$.*

(35) *m-words system can be generated by 3-words system for $m \geq 4$.*

Geometric understanding on 3-generation and reduction in m-words system

In order to understand on the 3-generation we will treat it by use of geometry of binary and ternary structures. We begin with the geometry of the binary. By use of the association we can give the geometric understanding on the 3-generation:

3-generation is basic

By use of the triangulation of polygons we can understand that the triangle is basic in our approach

Geometric understanding on 3-generation

We can express the 3-generation of m-words system as in Fig. 14 manner:

The association for higher degree polygons is completely similar.

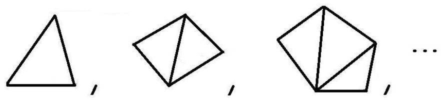

Fig. 13: The triangle is basic in our approach.

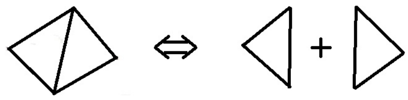

Fig. 14: $\mathcal{L}^{(4)} \Longleftrightarrow \mathcal{L}'^{(3)} \cap \mathcal{L}''^{(3)}$.

V. Turning machine structures in particle physics

In this section we associate the field theory to the Turning machine. We shall find the evolution structure in particle physics and give origins of "the binary structure of mesons" and "the ternary structure of baryons". Also we can show that a single quark is confined at present and m-quarks ($m \geq 4$) can be observed only in the resonance of 2,3-quarks [2]. This will be discussed in Part II of the paper which will appear separately.

1. Push-down automaton which is given by a bosonic field

At first we give an example which supports our basic idea on the association. We choose a bosonic algebra which is generated by creation operator a^* and annihilation operator a satisfying the following commutation relation:

$$a^* - a^* a = 1.$$

The bosonic field can be constructed by operating the creation operator in vacuum $\langle 0|$:

$$\begin{aligned}
\Psi(z) &= \langle 0|a^*, \\
\Psi^2(z) &= \Psi(z_1) \otimes \Psi(x_2) = \langle 0|a^* a^* = \Psi(x_1)a^*, \\
&\ldots, \\
\Psi^n(z) &= \Psi^{n-1}(x)a^*, \\
&\ldots,
\end{aligned}$$

Next we consider the conjugate vacuum $|0\rangle$ which satisfies the following conditions:

$$\begin{aligned}
\Psi^*(\kappa_1) &= a|0\rangle, \\
\Psi^{*2}(\kappa) &= a\Psi^*(\kappa_1) = a^2|0\rangle, \\
&\dots, \\
\Psi^{*n}(\kappa) &= a\Psi^{*n-1}(\kappa) = a^n|0\rangle, \\
&\dots,
\end{aligned}$$

Here we assume that

(36) $\langle 0|0\rangle = 1,$

(37) $\langle 0|a\rangle = 0,\ a^*|0\rangle = 0.$

Then we can construct the bosonic field as follows:

$$\Psi(x_1) \otimes \Psi(x_2) \otimes \ldots \otimes \Psi(x_h)$$

is an n-particle field. Then we have

$$\begin{aligned}
a^*\Psi(x_1) \otimes \ldots \otimes \Psi(x_n) &= \Psi(x_1) \otimes \ldots \Psi(x_{n+1}), \\
a\Psi(x_1) \otimes \ldots \otimes \Psi(x_n) &= \Psi(x_1) \otimes \ldots \otimes \Psi(x_{n-1}).
\end{aligned}$$

When we consider a Hamiltonian operator then we can consider the quantum bosonic field in physics. We can prove the following

Proposition 8. *By the following associations we can obtain the push-down automaton* (Fig. 15):

1. $\langle o| \implies$ ¢
2. $|o\rangle \implies \$$
3. $\Psi(x_1) \otimes \cdots \otimes \Psi(x_n) \implies$

¢	a_1	a_2	$\cdots\cdots$	a_n		$\$$

4. $a^*\Psi(x_1) \otimes \cdots \otimes \Psi(x_n) \implies$

¢	a_1^*	$\cdots\cdots$	a_n^*	a_{n+1}^*	$\cdots$	$\$$

5. $a\Psi(x_1) \otimes \cdots \otimes \Psi(x_n) \implies$

¢	a_1^*	$\cdots\cdots$	a_{n-1}^*		$\cdots$	$\$$

Fig. 15: The push-down automaton.

Here, associating the bracket:

$$\langle 0| \Rightarrow \;'' \subset'' \begin{pmatrix} \text{Push down} \\ \text{stack} \end{pmatrix}, \quad |0\rangle \Rightarrow \;'' \supset'' \begin{pmatrix} \text{Pop up} \\ \text{stack} \end{pmatrix}$$

we can obtain the push-down automaton. The acceptable sequences can be obtained by

$$\langle 0|X|0\rangle \neq 0.$$

2. The basic idea on associating the Turing machine to the field theory

We begin the symmetrization of the n-particle system and we shall find the association of these fields to $\mathcal{L}^{(m)}$. For this we consider the particle systems which are generated by m ($m \geq \varepsilon$) kinds:

$$\{a_1^*, a_1\}, \dots \{a_m^*, a_m\} :$$
$$\langle 0|a_{i_1}^{*n} \dots a_{im}^{*n} = \Psi_{i_1}(x_1)^n \dots \Psi_{im}(x_m)^n.$$

Next we consider the symmetrization of the system:

$$\langle 0|a_1^{*n} \dots a_m^{*n}|_S = \sum \frac{1}{m!} \langle 0|a_{i_1}^* \dots a_{i_m}^*.$$

Then we set

$$\langle 0|a_1^{*n} \dots a_m^{*n}|_S \Rightarrow a_1^{*n} \dots a_m^{*n},$$

and we obtain the desired association. Here we have to pay attention to the fields. For the case of bosonic fields the symmetrization causes no problem and its quite natural. For the case of fermionic field we have:

$$a_i^* a_j^* + a_j^* a_i^* = a_i a_j + a_j a_i = 2\delta_{ij}.$$

Hence we have to replace the symmetrization with the antisymmetrization:

$$\langle 0|a_i^* \dots a_n^{*n}|_A = \sum \frac{1}{m!} \sigma \begin{pmatrix} i_1 \dots i_m \\ 1 \dots m \end{pmatrix} \langle 0|a_{i_1}^* \dots a_{im}^*.$$

Taking these facts into account we associate the fields to the type of Turing machine.

3. The automaton type

We take a creation operator a^* and the one-side vacuum $\langle 0|$, and generate a field:

$$\langle 0|, \quad \langle 0|a^*(= \Psi_1), \quad \langle 0|a^{*2}(= a^*\Psi_1)$$
$$= \Psi_1(x) \otimes \Psi(x_2), \dots, \langle 0|a^{*N} = \Psi(x_1) \otimes \langle \Psi(x_N).$$

Then we can associate the automaton with the following transition diagram:

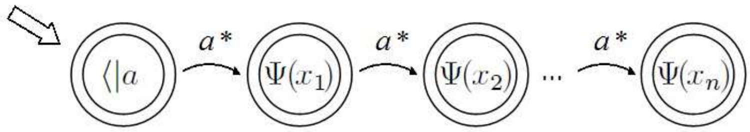

Fig. 16: The automaton type transition diagram.

We can generalize the field theory which is generated by several creation operators:

$$a_1^*, a_2^*, \ldots, a_m^*.$$

4. The push-down automaton type

As we have seen in Sect. I when we introduce the dual vacuum $|0\rangle$ and the annihilation operator we can define the push-down automaton. As we have seen we can identity the bracket sequence with this field:

$$a^* \Longleftrightarrow (, \qquad a \Longleftrightarrow).$$

Hence, we have for example:

$$a^*a^*a\ a^*aa \Longleftrightarrow ((.)(\ ))\,.$$

When we wish to generalize the field generated by a single element a^* and its conjugate element a to that which is generalized by several elements $a_1^*, \ldots, a_m^*, a_1, \ldots, a_m$. We may Dyke languages to these fields.

5. The linear bonded automaton

Next we proceed to the generation of baryons. At first we notice the basic facts on baryons:

(38) Any baryon constitutes three quarks.

(39) Any quark a color. The color constitutes the kinds G.R.Y. (Three colors).

(40) Baryons can be changed by week int-action and others (Fig. 17).

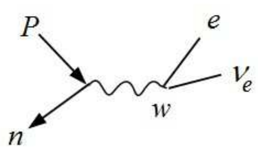

Fig. 17: Generation of baryons.

We shall show these properties by use of a linear bounded automaton. Usually we do not assume that the number of particles is bounded. But this is not acceptable from the cosmology because the size of the universe is finite at present; hence its total number is finite. Taking this fact into account we can int-reduce a linear bounded automaton. Namely, we consider the Turing machine which does not change its length we can consider the linear bounded automaton. This automaton can operate when the size of tapes is fixed. We need not produce a blank cell. We may understand that this system starts to "rearrange" by "interchange" or "mix" in the fixed particles.

6. General Turing machine

The particle system, which has a fixed number of particles and the universe (or ambient space) is stable, has the automaton of type (iii). When the universe spreads then we can find a new empty space and create a new particle. Then we may expect that a new Turing machine starts to operate. Taking the fact that Turing machines can describe the total finite mathematics into account, we can describe any particles which can not realize real physical particles.

Hence, *it might be quite natural that physical particles are not described by the general Turing machine.*

References

[1] M. Gell-Mann and Y. Ne'eman, *The Eight-Fold Way*, W. A. Benjamin, Inc., New York-Amsterdam 1964.

[2] K. Huang, *Quarks, Leptons and Gauge Fields*, 2nd ed. World Scientific, Singapore 2013.

[3] J. Ławrynowicz, O. Suzuki, and A. Niemczynowicz, *On the ternary approach to Clifford structures and Ising lattices*, Advances Appl. Clifford Algebras **22**, no. 3 (2012), 757–769.

[4] J. Ławrynowicz, O. Suzuki, and A. Niemczynowicz, *Fractals and chaos related to Ising-Onsager-Zhang lattices vs. the Jordan-von Neumann-Wigner procedures. Ternary approach*, Internat. J. of Nonlinear Sci. and Namer. Simul. **14**, no. 3–4 (2013), 211–215.

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STRUKTURY BINARNE I TERNARNE W FIZYCE I
STRUKTURA HIERARCHICZNA MASZYNY TURINGA W FIZYCE

S t r e s z c z e n i e

Przypominamy podstawowe fakty o maszynie Turinga i obserwujemy jej hierarchiczną strukturę. Przyporządkowując cząstkę i anty-cząstkę automatonowi obniżania, a trzy kwarki liniowemu ograniczonemu automatonowi, uzyskujemy model fizyki kwarków w terminach maszyny Turinga

Słowa kluczowe: struktura binarna, struktura ternarna, układy cząstek, pole bozonowe, typ automatonu obniżania, algebry ternarne i kubiczne

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Contribution to the jubilee volume, dedicated to Professors J. Ławrynowicz and L. Wojtczak

Krzysztof Warda, Daniel Baldomir, Manuel Pereiro, Juan E. Arias, Victor Pardo, and Jorge Botana

MAGNETORESISTANCE IN THIN FILMS INCLUDING THE DOMAIN STRUCTURE

Summary

We have shown that the magnetoresistance of the domain structure present in the thin layer of iron is dependent on the relative ratio width of the domain wall to the length of the sample. Moreover magnetoresistance as a function of the magnetic field has a similar behavior as GMR trilayer.

Keywords and phrases: magnetoresistance, thin film, domain structure, domain wall, magnetism

1. Introduction

The first clear experimental indication of magnetoresistance (MR) in ferromagnetic striped domain structure was observed by Allen and his coworkers [1] in a homogenous magnetic system. The authors investigated MR behavior in a thin layer of cobalt, considered as a function of an applied magnetic field. The authors emphasize the quantitative agreement between the domain walls and a trilayer.

In order to calculate MR in the domain structure we consider the organization of domains structure in thin films. In 1935 Landau and Lifschitz [2] showed that the existence of domains is a consequence of energy minimization. A single domain sample has associated with it large, magnetostatic energy, but the breakup of the magnetization into localized regions-domains, providing for flux closure at the ends of the specimen, reduces the magnetostatic energy. The decrease in magnetostatic energy is greater than the energy necessary to form magnetic domain walls, then multi-domain specimens will arise.

The thin film is divided into n monoatomic layers labeled by ν in the direction of y perpendicular to the surface [3]. The position of lattice site in the plane (x, z) is given by vector $\vec{j}$. We suppose that the film has dimensions L_x, L_y, L_z.

The experimental studies of the atomic spin structure of phase domain walls in the antiferromagnetic Fe monolayer on W (001) by means of spin-polarized scanning tunneling microscopy show that the domain wall width is only to 6–8 atomic rows [4]. This fact has significance from the point of view of feature applications.

Let us consider the direction of the easy axis lying in the plane of the sample. In the case where the uniaxial anisotropy is sufficiently large, we can expect that the domain structure observed experimentally has a form of stripe structure with the stripe magnetized parallel and antiparallel to easy magnetization axis. The Hamiltonian for the considered structure can be written as:

$$
\begin{aligned}
H = & \sum_{m\langle \nu j, \nu' j' \rangle} H\begin{pmatrix} \nu' j' \\ \nu j \end{pmatrix} c^{+}_{\nu j m} c_{\nu' j' m} + I \sum_{\nu j} c^{+}_{\nu j \uparrow} c_{\nu j \downarrow} \\
& + \frac{1}{2}(g\mu_B)^2 \sum_{\alpha\beta} \sum_{\nu j \nu' j'} N^{\alpha\beta}_{\nu j \nu' j'} S^{\alpha}_{\nu j} S^{\beta}_{\nu' j'} - K \sum_{\nu j} \langle S^{Z}_{\nu j} \rangle S^{Z}_{\nu j}.
\end{aligned} \tag{1}
$$

The Hamiltonian described by (1) includes four terms: The hopping term, the intra-atomic Coulomb interaction, the term corresponding to the magnetic dipolar interaction, and the energy of the uniaxial anisotropy. The symbol I denotes the intra-atomic Coulomb integral, while K represents the uniaxial anisotropy constant. Moreover, Hamiltonian (1) describes the magnetic properties for the uniaxial thin films magnetized homogenously. In order to find the effective Hamiltonian for thin film with the striped structure we use the new operators b^{+}_{rkm} and b_{rkm} by means of the following transformations:

$$
\begin{aligned}
c^{+}_{rkm} &= \sum_{m'} o^{+}_{rk} \begin{pmatrix} m' \\ m \end{pmatrix} b^{+}_{rkm'} \\
c_{rkm} &= \sum_{m'} o_{rk} \begin{pmatrix} m' \\ m \end{pmatrix} b_{rkm'}
\end{aligned} \tag{2}
$$

and for spin:

$$
S^{\alpha}_{rk} = \sum_{\alpha'} R^{\alpha\alpha'}_{rk} S^{\alpha'}_{rk} \tag{3}
$$

where the elements of $R^{\alpha\alpha'}_{rk}$ are the components of the rotation matrix. The operator O^{+}_{rk} is defined as:

$$
O^{+}_{rk} = \begin{pmatrix} e^{i\frac{\varphi_{rk}}{2}} \cos\frac{\vartheta_{rk}}{2} & e^{-i\frac{\varphi_{rk}}{2}} \sin\frac{\vartheta_{rk}}{2} \\ -e^{-i\frac{\varphi_{rk}}{2}} \sin\frac{\vartheta_{rk}}{2} & e^{i\frac{\varphi_{rk}}{2}} \cos\frac{\vartheta_{rk}}{2} \end{pmatrix}. \tag{4}
$$

Moreover, the following relation is fulfilled:

$$O_{rk}\binom{m'}{m} = \left(O^{+}_{rk}\binom{m'}{m}\right)^{*}. \tag{5}$$

The angles φ_{rk} and ϑ_{rk} correspond to the rotations in spherical coordinates, and φ_{rk} to the rotation in the plane of the layer, while ϑ_{rk} is responsible for deviation of local z axis with respect to the global system. The transformations lead to the following Hamiltonian for thin film with domain structure [5]:

$$\begin{aligned} H \;=\; & \sum_{\nu j,\nu' j'} H\binom{\nu' j'}{\nu j} c^{+}_{\nu jm} c_{\nu' j' m}\left[\left(\sum_{m} b^{+}_{\nu jm} b_{\nu' j' m}\right)\cos\frac{\vartheta_{\nu j}-\vartheta_{\nu' j'}}{2}\right. \\ & \left. +(b^{+}_{\nu j\uparrow} b_{\nu' j'\downarrow} - b^{+}_{\nu j\uparrow} b_{\nu' j'\downarrow})\sin\frac{\vartheta_{\nu' j'}-\vartheta_{\nu j}}{2}\right] \\ & +I\sum_{\nu j} n_{\nu j\uparrow} n_{\nu j\downarrow} + \frac{1}{2}(g\mu_B)^2 \sum_{\alpha\beta}\sum_{\nu j\nu' j'} N^{\alpha\beta}_{\nu j\nu' j'} R^{\alpha\alpha'}_{\nu j} R^{\beta\beta'}_{\nu' j'} {S'}^{\alpha'}_{\nu j} {S'}^{\beta'}_{\nu' j'} \\ & -K\sum_{\nu j}\langle {S'}^{Z}_{\nu j}\rangle \cos\vartheta_{\nu j}\left({S'}^{Z}_{\nu j}\cos\vartheta_{\nu j} + {S'}^{x}_{\nu j}\sin\vartheta_{\nu j}\right) \end{aligned} \tag{6}$$

where:

$$n_{\nu j\uparrow} = b^{+}_{\nu jm} b_{\nu jm}. \tag{7}$$

For the rotation around the y axis perpendicular to the surface of the film, we can write:

$$R_{\nu j} = \begin{pmatrix} \cos\vartheta_{\nu j} & 0 & -\sin\vartheta_{\nu j} \\ 0 & 1 & 0 \\ \sin\vartheta_{\nu j} & 0 & \cos\vartheta_{\nu j} \end{pmatrix}. \tag{8}$$

In the molecular field approximation, the Hamiltonian (6) can be rewritten as:

$$\begin{aligned} \langle H\rangle \;=\; & \langle H_0\rangle + (g\mu_B)^2 \sum_{\nu j\nu' j'} N^{\alpha\beta}_{\nu j\nu' j'} \langle {S'}^{Z}_{\nu j}\rangle\langle {S'}^{Z}_{\nu' j'}\rangle - K\sum_{\nu j}\langle {S'}^{Z}_{\nu' j'}\rangle^2 \cos^2\vartheta_{\nu j} \\ & + \sum_{\nu j,\nu' j'} H\binom{\nu' j'}{\nu j}\left[\left(\sum_{m}\langle b^{+}_{\nu jm} b_{\nu' j' m}\rangle\right)\cos\frac{\vartheta_{\nu j}-\vartheta_{\nu' j'}}{2}\right. \\ & \left. + \left(\langle b^{+}_{\nu j\uparrow} b_{\nu' j'\downarrow}\rangle - \langle b^{+}_{\nu j\downarrow} b_{\nu' j'\uparrow}\rangle\right)\sin\frac{\vartheta_{\nu' j'}-\vartheta_{\nu j}}{2}\right]. \end{aligned} \tag{9}$$

In the last expression, $\langle H_0\rangle$ represents the part of Hamiltonian H which is not dependent on the angle of $\vartheta_{\nu j}$, moreover:

$$\langle b^{+}_{\nu jm} b_{\nu' j' m}\rangle = \sum_{\tau h} T^{m}_{\nu j\tau h} T^{m}_{\nu' j'\tau' h'} \langle b^{+}_{\tau hm} b_{\tau hm}\rangle \delta_{mm'} \tag{10}$$

and

$$N_{\nu j\nu' j'} = \sum_{\alpha\beta} N^{\alpha\beta}_{\nu j\nu' j'} R^{\alpha z}_{\nu j} R^{\beta z}_{\nu' j'}. \tag{11}$$

2. Magnetoresistance of striped structure in thin film

In order to find the magnetoresistance in thin layer, we consider a thin magnetic film in which the angle distribution of magnetization is homogenous. In first approximation the domain wall is omitted and the domains are treated as ellipsoids (Fig. 1).

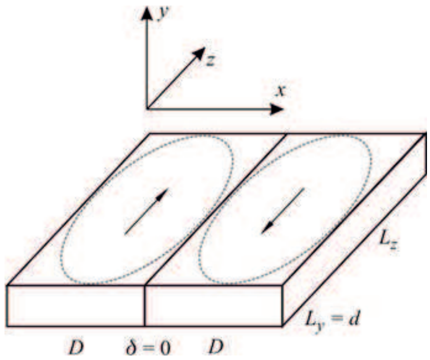

Fig. 1: Domain structure of a magnetic thin film with the 0 width of domain wall.

The mean value of the Hamiltonian given by Eq. (9) for thin layer can be rewritten in the form of:

$$\langle H \rangle = \langle H_0 \rangle + \frac{V}{4L_r} \int_0^{L_r} L(\vartheta(r))dr, \tag{12}$$

where $\langle H_0 \rangle$ represents part of $\langle H \rangle$ which is independent of the angle ϑ in the plane of the layer. The volume of the considered sample is equal:

$$V = Na^3 = N_x N_y N_z a^3, \tag{13}$$

where a is the lattice constant, N is a number of lattice sides, while $L_r = aN_r$ represents the size of layer in direction r ($r = x$ or y). The functional $L(\vartheta(r))$ has form:

$$L\left(\vartheta(r)\right) = \left(\alpha \left(\frac{d\vartheta}{dr} \right)^2 + \frac{K}{a^3} \sin^2 \vartheta + \frac{g\mu_B^2}{a^6} \sum_{j'r} N_{\nu j \nu' j'} \right), \tag{14}$$

where

$$N_{\nu j \nu' j'} = \sum_{\alpha\beta} N^{\alpha\beta}_{\nu j \nu' j'} R^{\alpha z}_{jr} R^{\beta z}_{jr}. \tag{15}$$

The mean value of the Hamiltonian can be obtained by taking into account demagnetized field generated by domains. In case of ellipsoidal domains described here, the demagnetizing coefficients are as follows:

$$N^{xx} = 0, \quad N^{yy} = 0, \quad N^{zz} = \frac{\pi^2}{2V} \frac{DL_y}{L_z^2}, \tag{16}$$

where D is the width of domain, and moreover the following conditions are fulfilled:

$$\cos^2\vartheta(r) = 1, \quad \sin^2\vartheta(r) = 0. \tag{17}$$

The functional $L(\vartheta(r))$ can be rewritten as:

$$L(\vartheta(r)) = \left(\alpha\left(\frac{d\vartheta}{dr}\right)^2 + \frac{g\mu_B^2}{a^3}VN^{zz}\right). \tag{18}$$

And taking into account relation (16), we obtain:

$$L(\vartheta(r)) = \left(\alpha\left(\frac{d\vartheta}{dr}\right)^2 + \frac{\pi^2(g\mu_B)^2 DL_y}{2a^3L_z^2}\right). \tag{19}$$

If in the direction x, the sample includes s domains, then the mean value of the Hamiltonian can be written as:

$$\langle H\rangle = \langle H_0\rangle + \frac{V}{4L_x}\left(\int_0^{L_x}\alpha\left(\frac{d\vartheta}{dx}\right)dx + \int_0^{L_r}\frac{\pi^2(g\mu_B)^2DL_y}{2a^3L_z^2}dx\right). \tag{20}$$

The energy of the domain on the unit area of domain wall s depends on the angle distribution of magnetization and can be expressed as:

$$\sigma = \frac{\alpha}{s}\int_0^{L_r}\left(\frac{d\vartheta}{dr}\right)^2 dr. \tag{21}$$

Introducing the last relation in 16, we obtain in the considered case:

$$\langle H\rangle = \langle H_0\rangle + \frac{N}{4}\left(-\frac{a^3}{D^2}\sigma + \frac{\pi^2(g\mu_B)^2L_y}{2a^3L_z^2}\right). \tag{22}$$

In order to calculate D, we minimalize the last equation and put $d\langle H\rangle/dD = 0$, this leads to the results for D:

$$D = \frac{\pi^2(g\mu_B)^2L_y}{2a^3L_z^2}. \tag{23}$$

The last relation allows us to estimate the critical thickness of layer for which appears in the domain. For $L_x = L_y$ and $D = L_z$ we obtain:

$$L_y^{\mathrm{crit}} = \frac{2}{\pi^2}\left(\frac{a^3}{g\mu_B}\right)^2\sigma. \tag{24}$$

3. An influence of the external magnetic field

When we applied an external magnetic field to the domain structure given on Fig. 2a, in first approximation, we suppose the thickness of the domain wall equals 0. The thickness of domains is denoted by D^+ and D^-. The thickness of D^+ and D^- can expand or narrow under the influence of an external magnetic field H in direction z. When the magnetic field is 0, then $D^+ = D^-$. When the critical magnetic field is applied, the domain wall is shifted to the edge of the sample, and we have $D^- = 0$ and $D^+ = 2D$ (Fig. 2b).

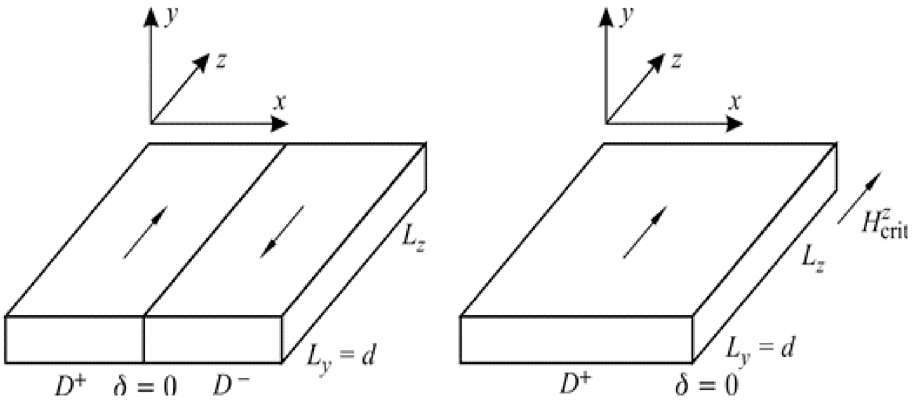

Fig. 2: The influence of an external magnetic field on the domain structure: a) without external magnetic field; b) when the critical magnetic field is applied.

The influence of an external magnetic field on the domain structure was considered in many papers [6–8]. Here we consider the influence of a longitudinal external magnetic field applied to a thin film with the stripe domain structure. The calculations procedure used for the bulk structure [2] can applied in this case. The Hamiltonian should be completed by a term describing the interaction of the spins in own system, with the external field H^z which is the Zeeman type field, and can be written in the form of:

$$\langle H_H \rangle = -\frac{1}{2} g\mu_B H^z \cos\vartheta(x) \tag{25}$$

and the functional $L(\vartheta(x))$ is:

$$\begin{aligned} L(\vartheta(x)) &= \alpha \left(\frac{d\vartheta}{dx}\right)^2 + \frac{K}{a^3}\sin^2\vartheta(x) + \frac{(g\mu_B)^2}{a^3} N N^{zz} \cos^2\vartheta(x) \\ &\quad - \frac{g\mu_B H^z}{2a^3}\cos\vartheta(x). \end{aligned} \tag{26}$$

Taking into account the conditions (16), we obtain the equation for the determination of the demagnetization coefficients:

$$\frac{(g\mu_B)^2}{a^3} N N^{zz} = \frac{(g\mu_B)^2}{a^3} N \frac{\pi^2}{2V}\frac{L_y}{L_z^2} D^{\pm} = \left(\frac{g\mu_B}{a^3}\right)^2 \frac{\pi^2}{2}\frac{L_y}{L_z^2} D^{\pm}. \tag{27}$$

Minimizing the energy of functional given by (26), we obtain:

$$\begin{aligned} L &= \frac{1}{L_x}\int_0^{L_x} L(\vartheta(x))dx = \frac{\sigma}{D} + \frac{1}{2}\sigma_0 \frac{L_y}{L_z^2}\frac{(D^+)^2 - (D^-)^2}{2D} \\ &\quad - q\frac{(D^+)^2 - (D^-)^2}{4D^2}, \end{aligned} \tag{28}$$

where the parameters in (28) are given as:

$$
\begin{cases}
\sigma_0 = \pi^2 \left(\dfrac{g\mu_B}{a^3}\right)^2, \\[2ex]
q = \dfrac{1}{2} g\mu_B \dfrac{H^z}{a^3}.
\end{cases} \tag{29}
$$

As we can introduce $D^{\pm} = D(1 \pm \varepsilon)$ and substitute this term in Eq. 28, we can obtain:

$$L = \frac{\sigma}{D} + \frac{1}{2}\sigma_0 \frac{L_y}{L_z^2}\left(D + \varepsilon^2 D\right) - q\varepsilon. \tag{30}$$

In order to obtain the value of ε, we minimalize the last expression with respect to ε:

$$\frac{dL}{d\varepsilon} = \sigma_0 \frac{L_y}{L_z^2} D\varepsilon - q. \tag{31}$$

And putting $dL/d\varepsilon = 0$, we finally obtain:

$$\varepsilon = \frac{1}{2\pi^2}\left(\frac{H^z}{g\mu_B}\right)\frac{L_z^2 a^3}{L_y D}. \tag{32}$$

From last expression it is easy to obtain the value of a critical field. When the critical field is applied, then $D^- = 0$ and $D^+ = 2D$, and this fact leads to the value of critical field given as:

$$H^z_{\mathrm{crit}} = \frac{2\pi^2 g\mu_B L_y D}{L_z^2 a^3}. \tag{33}$$

4. Magnetoresistance of strip structure in thin films

In order to find the magnetoresistance in a thin layer, we consider a thin magnetic film in which the angle distribution of magnetization is homogenous. The systems composed of FM layers and an NM spacer correspond to the stripe domain structure.

The experimental picture of the observed structure can be represented in a model of parallel domains with variable magnetization [9]. In Fig. 3 the model structure of stripe domains is presented with a width D and the width of the domain walls equal δ, cut into direction of n in the domain structure.

In order to find MR, let us consider the stripe domain structure with zero width of domain walls, through which the current flows perpendicular to the magnetization. Moreover, the magnetic field is applied in the plane (see Fig. 4).

In an analogy to the description of GMR in trilayers, we used the model of the resistor network which corresponds to different configuration of spin-up and spin-down electrons, and the width of the domain. We consider the case where the magnetization is parallel to the applied magnetic field. The width of the domain expanding under an influence of the magnetic field is denoted by D^+, while the domain narrowing is denoted by D^-. The value of the resistors shown in Fig. 4b is given as:

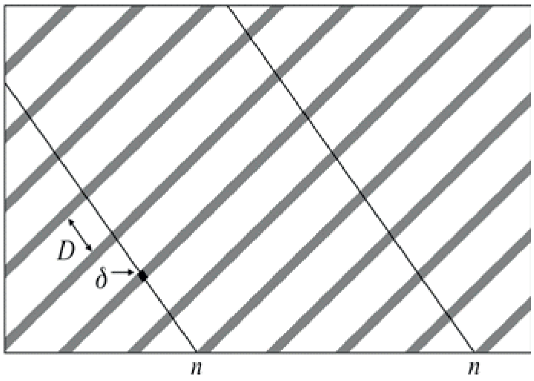

Fig. 3: The model representation of the stripe domain structure.

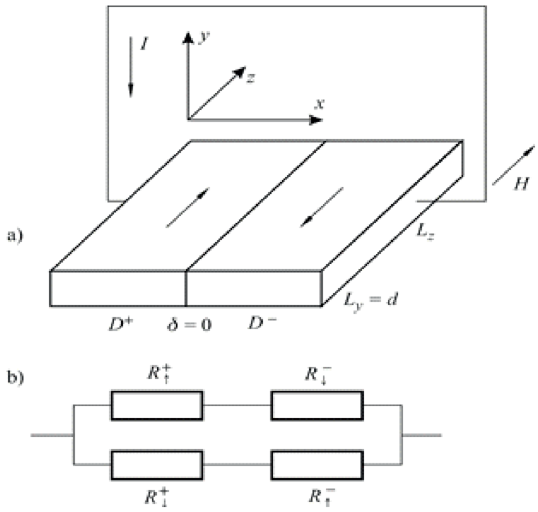

Fig. 4: a) Domain structure through which the current flows; b) the equivalent resistor network of the considered system.

(34)
$$R_\uparrow^+ = \rho_\uparrow \frac{D^+}{dL_z}, \quad R_\downarrow^+ = \rho_\downarrow \frac{D^+}{dL_z},$$
$$R_\uparrow^- = \rho_\uparrow \frac{D^-}{dL_z}, \quad R_\downarrow^- = \rho_\downarrow \frac{D^-}{dL_z},$$

The total resistance of network in Fig. 4b can be obtained as:

(35)
$$\frac{1}{R_{\uparrow\downarrow}} = \frac{1}{R_\uparrow^+ + R_\downarrow^-} + \frac{1}{R_\downarrow^+ + R_\uparrow^-}$$

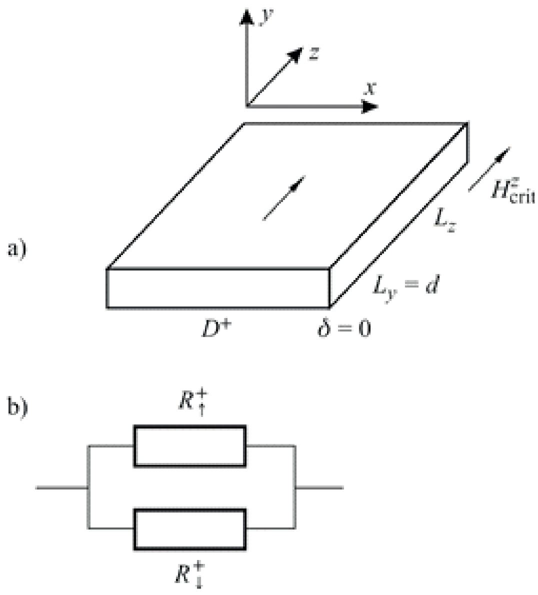

Fig. 5: a) The domain structure where the critical magnetic field is applied; b) the corresponding resistor network.

and taking into account the relations (34) is equal:

$$\frac{1}{R_{\uparrow\downarrow}} = \frac{dL_z}{D^+\rho_\uparrow + D^-\rho_\downarrow} + \frac{dL_z}{D^+\rho_\downarrow + D^-\rho_\uparrow}. \tag{36}$$

Taking into account the relation $D^\pm = D(1 \pm e)$ we obtain:

$$\frac{1}{R_{\uparrow\downarrow}} = \frac{dL_z}{D}\ \frac{2(\rho_\uparrow + \rho_\downarrow)}{[(\rho_\uparrow + \rho_\downarrow) + \varepsilon(\rho_\uparrow - \rho_\downarrow)]\ [(\rho_\uparrow + \rho_\downarrow) - \varepsilon(\rho_\uparrow - \rho_\downarrow)]}. \tag{37}$$

Denoting:

$$\Delta\rho = \frac{\rho_\uparrow - \rho_\downarrow}{\rho_\uparrow + \rho_\downarrow}, \tag{38}$$

the relation (37) can be rewritten as:

$$\frac{1}{R_{\uparrow\downarrow}} = \frac{dL_z}{D}\ \frac{2(\rho_\uparrow + \rho_\downarrow)}{(\rho_\uparrow + \rho_\downarrow)^2(1 - \varepsilon^2(\Delta\rho)^2)} = \frac{dL_z}{D}\ \frac{2}{(\rho_\uparrow + \rho_\downarrow)(1 - \varepsilon^2(\Delta\rho)^2)}. \tag{39}$$

Another interesting case should be considered – one where the critical magnetic field is applied (see Fig. 5) and the domain wall is shifted on the edge of the sample, and the value of resistance for this case can be given as:

$$\frac{1}{R_{\uparrow\uparrow}} = \frac{1}{\frac{2D}{dL_z}\rho_\uparrow} + \frac{1}{\frac{2D}{dL_z}\rho_\downarrow} = \frac{dL_z}{D}\ \frac{(\rho_\uparrow + \rho_\downarrow)}{2}\ \frac{1}{\rho_\uparrow\rho_\downarrow}. \tag{40}$$

The value of MR we defined as:

$$MR = \frac{R(H)}{R(H=0)} \tag{41}$$

and leads to the following results:

$$MR = \frac{R(H)}{R(H=0)} = 1 - \varepsilon^2(\Delta\rho)^2. \tag{42}$$

The behavior of MR as a function of an applied magnetic field for Fe is presented in Fig. 6.

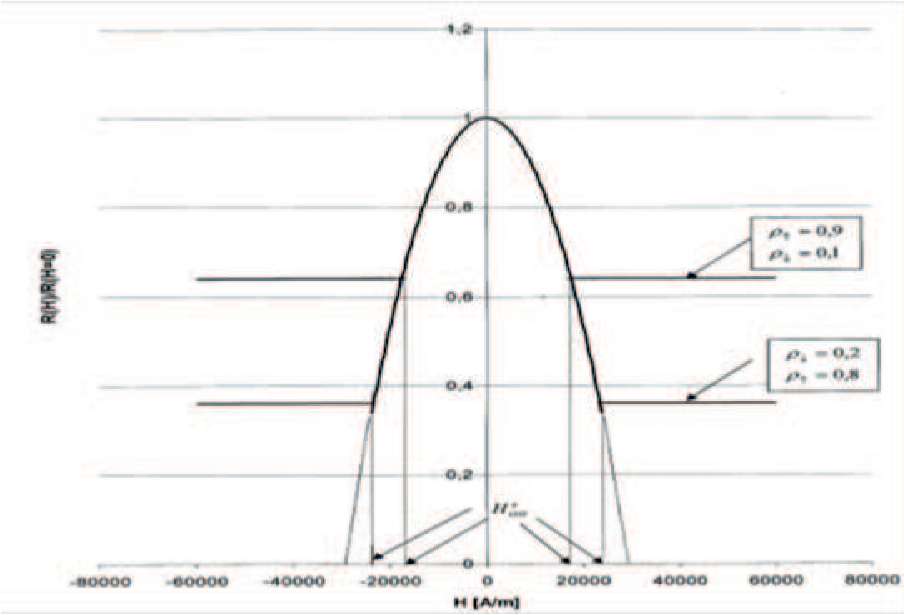

Fig. 6: Magnetoresistance of Fe versus an applied magnetic field.

The result obtained for MR is analogical to the case of a trilayer comprising two FM layers separated by a NM film. The behavior of GMR in multilayers was investigated experimentally by P. Grünberg [6] and A. Fert [5]. The experimental curve of GMR exhibits similar behavior as a magnetic field is applied to the sample.

5. MR of the stripe domain structure with domain wall of thickness δ

In the next part, we find the MR of the stripe domain structure, with domain wall of thickness δ, in which the current flows perpendicularly to the magnetization. The schematic illustration of the considered system is presented in Fig. 7.

By analogy to our earlier considerations, D^+, D^- are the thickness of domain walls with different orientation of magnetization, while D_W is the thickness of domain wall at the border between domains D^+ and D^-. The schematic resistor network is presented in Fig. 8.

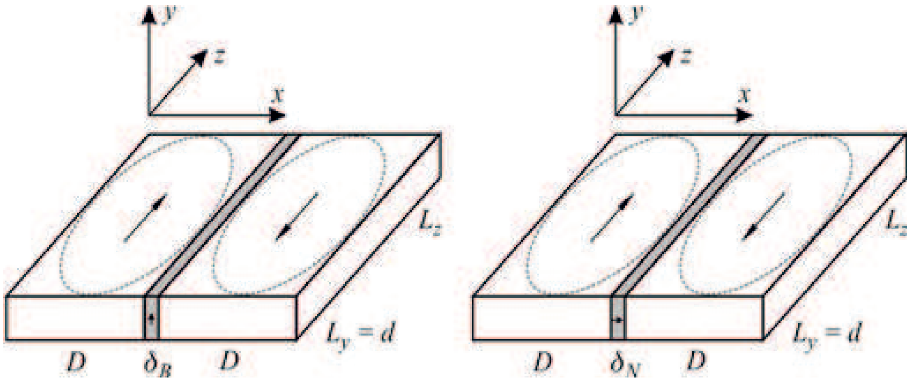

Fig. 7: Domain structure with none-zero domain wall for two different types of walls: a) Bloch wall and b) Néel wall .

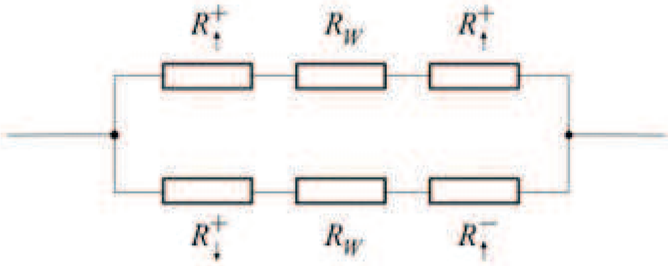

Fig. 8: The resistor network for two domains separated by a domain wall for an antiparallel configuration of two adjacent domains (R_W is a resistance of the domain wall).

In the light of our earlier considerations for two domains with different magnetic moment orientations, the resistance for the considered system can be introduced as:

$$\frac{1}{\rho_{\uparrow\downarrow}} = \frac{D}{\rho_\uparrow(D + D^+) + \rho_W D_W + \rho_\downarrow D^-} + \frac{D}{\rho_\downarrow(D + D^+) + \rho_W D_W + \rho_\uparrow D^-}, \tag{43}$$

which can be rewritten as:

$$\frac{1}{\rho_{\uparrow\downarrow}} = \frac{1}{\rho_\uparrow(1+\varepsilon) + \rho_W \frac{D_W}{D} + \rho_\downarrow(1-\varepsilon)} + \frac{1}{\rho_\downarrow(1+\varepsilon) + \rho_W \frac{D_W}{D} + \rho_\uparrow(1-\varepsilon)}. \tag{44}$$

For parallel orientation of magnetic moments in domains, we have:

$$\frac{1}{\rho_{\uparrow\uparrow}} = \frac{1}{2\rho_\uparrow + \rho_W \frac{D_W}{D}} + \frac{1}{2\rho_\downarrow + \rho_W \frac{D_W}{D}}, \tag{45}$$

$$\rho_{\uparrow\uparrow} = \frac{4\rho_\uparrow\rho_\downarrow + \rho_W^2 \frac{D_W^2}{D} + 2(\rho_\uparrow + \rho_\downarrow)\rho_W \frac{D_W}{D}}{2\left((\rho_\uparrow + \rho_\downarrow) + \rho_W \frac{D_W}{D}\right)} \left(\rho_\uparrow + \rho_\downarrow + \rho_W \frac{D_W}{D}\right). \tag{46}$$

The value of MR is given by:

$$MR = 1 - \frac{\rho_{\uparrow\uparrow}}{\rho_{\uparrow\downarrow}} = \frac{R(H)}{R(H=0)} \tag{47}$$

and finally this leads to the result:

$$MR = 1 - \frac{(\rho_\uparrow - \rho_\downarrow)^2 \varepsilon^2}{\left(\rho_\uparrow + \rho_\downarrow + \rho_W \frac{D_W}{D}\right)^2}. \tag{48}$$

The calculations of MR versus an applied magnetic field for domain structure comprising two domains separated by domain wall are shown in Fig. 9. For calculations we used the following values of parameters: the thickness of layer $L_y = 10\,\text{nm}$, the length of sample $L_z = 4\,\mu\text{m}$.

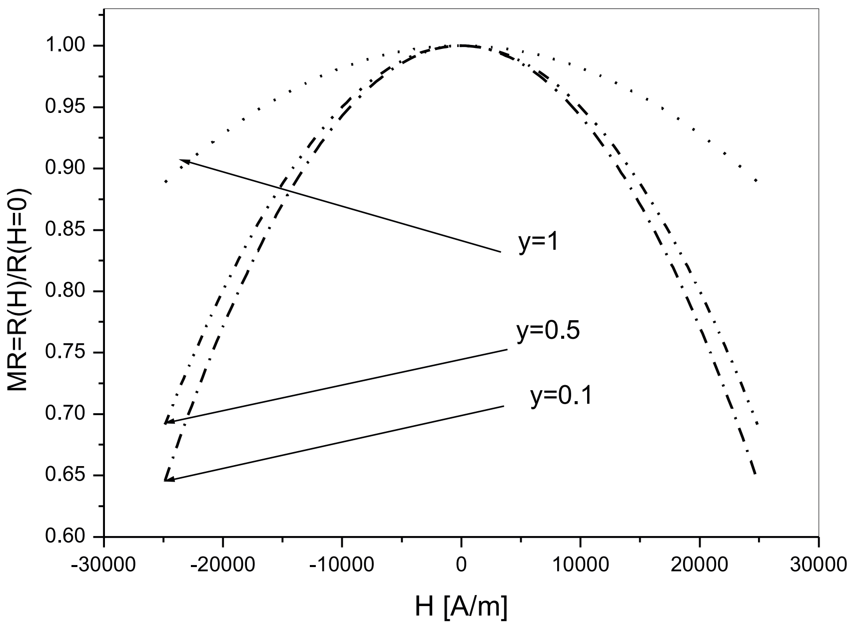

Fig. 9: MR for a structure including two domains separated by a domain wall versus an applied magnetic field for 3 selected values of parameter $y = \frac{\rho_W}{\rho_\uparrow} \frac{D_W}{D}$.

An interesting behavior of the Fe layers was observed in Zr/Fe/Zr trilayer structures [10] which behave as an array of ferromagnetic, interacting Fe clusters, below a thickness around 16 nm. Moreover, the considered trilayer exhibits the thickness dependence of the room temperature coercivity, which can be interpreted as evidence for a switch from a Bloch domain wall structure to a Néel interacting cluster behavior, as the Fe layer thickness is decreased below 16 nm. This effect can be related to the appearance of an Fe-Zr amorphous phase at the grain boundaries which influence progressively on the magnetization process when the magnetic layer thickness is reduced.

The evolution of the domain structure was investigated experimentally by means of Polar Kerr Microscopy, in a perpendicularly magnetized, Fe/Au multilayer structure, showing strong dependence between the character of the transition with the interlayer coupling type and the orientation of magnetization in the adjacent sublayers [11].

It was demonstrated experimentally and theoretically that DW motion in ferromagnetic films, with perpendicular anisotropy grown on a stepped substrate [12] can be tuned by modifying the underlying step density of the supporting substrate. It is worth of emphasizing that the dynamics of magnetic domain walls in such films appear to be spatially anisotropic and strongly dependent on the step density. The domain wall velocity in very thin layers is controlled by the exchange interaction of domain walls. This fact makes this phenomenon interesting for magnetic nanodevices.

Moreover, it is worth of emphasizing here that the ultrathin magnetic films with uniaxial anisotropy discussed here have been intensively studied as materials promising for applications, e.g. as magnetic storage media with perpendicular recording of information [13].

References

[1] W. Allen, J. F. Gregg, K. Ounadjela, M. Viret, M. Hehn, S. M. Thompson, and J. Coey, J. Mag. Mag. Mat. **165**, no. 1–3 (1997), 121–124.

[2] L. D. Landau and E. M. Lifschitz, Physik Z. Sovjetunion **8** (1935), 153.

[3] S. Sukiennicki and L. Wojtczak, Rev. Roum. Phys. **14** (1969), 1227.

[4] M. Bode, E. Y. Vedmedenko, K. von Bergmann, A. Kubetzka, P. Ferriani, W. Heinze, R. Wiesendanger, Nature Materials **5, 6** (2006), 477–481.

[5] L. Wojtczak and A. Sukiennicki, Acta Phys. Polon. A **51** (1977), 325.

[6] J. Klamut, Acta Phys. Polon. A **38** (1972), 873.

[7] W. Wasilewski and W. Ziętek, Acta Phys. Polon. A **42** (1972), 207.

[8] R. Świrkowicz, Acta Phys. Polon. A **49** (1976), 503.

[9] R. P. Ferrier, J. N. Chapman, and N. Toms, J. Vac. Sci. & Tech. **10** (1973), 640.

[10] F. J. Castaño, T. Stobiecki, M. Gibbs, M. Czapkiewicz, J. Wrona, and M. Kopcewicz, Thin Solid Films **348**, no. 1–2, 6 (1999), 233.

[11] M. Żołądź, T. Ślęzak, D. Wilgocka-Ślęzak, N. Spiridis, J. Korecki, T. Stobiecki, K. Röll, J. Mag. Mag. Mat. **272–276** (2004), 1253.

[12] A. Stupakiewicz, E. Y. Vedmedenko, A. Fleurence, T. Maroutian, P. Beauvillain, A. Maziewski, and R. Wiesendanger, Phys. Rev. Lett. **103** (2009), 137202.

[13] J. Kisielewski, A. Maziewski, K. Postava, A. Stupakiewicz, A. Petroutchik, L. T. Baczewski, and A. Wawro, J. Mag. Mag. Mat. **322** (2010), 1475.

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MAGNETOOPÓR W CIENKICH WARSTWACH ZAWIERAJĄCYCH STRUKTURĘ DOMENOWĄ

S t r e s z c z e n i e

Wykazaliśmy, iż magnetoopór struktury domeny wystepującej w cienkiej warstwie żelaza zależy od względnego stosunku szerokości ściany domeny oraz wielkości próbki. Ponadto pokazano, że magnetoopór w funkcji przyłożonego zewnetrznego pola magnetycznego ma podobne zachowanie jak GMR w układzie trójwarstwy.

Słowa kluczowe: magnetoopór, cienka warstwa, struktura domenowa, ściana domenowa, magnetyzm

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Contribution to the jubilee volume, dedicated to Professors J. Ławrynowicz and L. Wojtczak

Stamatia Artemi, Anthoula Maidou, Nikolaos Dintsios, and Hariton M. Polatoglou

PHYSICAL PROCESSES OF ENERGY TRANSFER RELATED TO EVERYDAY LIFE EXPERIENCES

Summary

This article describes several approaches to physical processes of energy transfer in the context of everyday life experiences that can be used in the teaching process of this field focusing on sustainable development. To achieve this, a polymorphic educational framework was developed, which includes a platform that hosts numerous educational tools, as simulations, real remote experiments, presentations, etc., that a teacher or any interested person can use. Simulations can be useful in modeling a physical phenomenon in different scales at the same time and they can help users better understand the theoretical structure of a phenomenon and also its applications in real life.

Keywords and phrases: energy transfer, Navier-Stokes equations, modelling vs. simulation

1. Introduction

1.1. Sustainable Development (SD)

The great changes of the last decades have shown the need for adapting developments that takes into account social and environmental issues. These kinds of developments were defined as sustainable. One of the generally accepted definitions of sustainability is "a development that meets the needs of the present without compromising the ability of future generations to meet their own needs" (WCED, 1987). Sustainability as an academic field has become a basic structure in university curricula. It has been integrated in academic programs from Harvard and Arizona State University to the

Technical University of Catalonia and the University of Tokyo (Wick, Withycombe & Redman, 2011). *Agenda 21* is the document resulting from the United Nations Conference on Environment and Development (UNCED), also called the Earth Summit, held in Rio de Janeiro, Brazil in 1992. As McKeown and Hopkins (2003) emphasize, "Agenda 21 calls for education in every chapter. Chapter 36 of Agenda 21, 'Promoting Education, Public Awareness, and Training', specifically identifies four major thrusts: (1) improving the quality of and access to basic education, (2) reorienting existing education to address sustainable development, (3) developing public understanding and awareness, and (4) training. In addition, sustainability, and therefore ESD, is commonly thought to involve and address three realms: environment, society, and economy". Charles Hopkins was involved in drafting Chapter 36 of Agenda 21. He noticed a core shift in intent between The UN Conference on the Human Environment (held in Stockholm in 1972) and the Rio conference (McKeown & Hopkins, 2003). The overall intent had shifted from the protection of the environmental and the reduction of pollution to encounter the needs of the environment and the society.

Education for Sustainable Development (ESD) has become a very important issue recently in the education of students worldwide, because it offers knowledge, skills, attitudes and values which are necessary to ensure a sustainable future for humanity at local and global levels. The decade 2005–2014 was named by the United Nations "Decade of Education for Sustainable Development" and UNESCO took a leading role in this effort. The scarcity of the major resources as food, energy, clean water etc. forces for measures to ensure the sustainability of life on our planet.

Sustainable Development (SD) refers to a development, which meets the current needs of the society without compromising the needs of future generations (World Commission on Environment and Development (WCED 1987, p43). This definition recognizes the importance of the environment and the need of society for growth (McKeown & Hopkins, 2002). SD is trying to create sustainability of the environment, the society and the economy. The importance of education for SD has been pointed out by bodies such as UNESCO (2002, 2004). UNESCO envisioned a world where everyone has the opportunity to benefit from education and acquire the values, behaviors and lifestyles that are necessary for a sustainable future and positive social transformations (UNESCO 2004, p4). In addition, education was a priority for the "Strategy for Education for Sustainable Development" (UNECE 2005), because it can develop and strengthen knowledge, skills and values that will enhance people of all ages to assume responsibility for creating and gaining a sustainable future (ibid). Although SD is a concept that has been discussed widely, it can be accessed and interpreted in many ways, depending on one's viewpoint, such as economic activity, technological approach or the relationship between the communities and the general economic context (Huckle 1996; O'Riordan & Voisey 1998; Fien & Tilbury, 2002). This variety of approaches leads to a number of trends and paradoxes reflected in the concepts and pedagogy of ESD (Scott & Gough 2003, Scott 2005). ESD is not just an innovation that can be adapted to the educational system, but a continuing process of pedagogical transformation (Fullan & Ballew 2001). ESD is a very important issue for the education of pupils worldwide because it offers knowledge, skills, attitudes and values necessary to ensure a sustainable future for humanity at the local and global level.

ESD includes Environmental Education (EE) but is essentially a broader approach (Reid 2002, McKeown & Hopkins 2003). ESD has its roots in EE, but goes beyond it (Bolscho & Hauenschild 2006), including issues related to reducing risks from disasters, cultural diversity, poverty reduction, gender equality issues, health promotion, peace and security and sustainable urbanization (UNESCO, 2004). There is plenty of contemporary literature on EE or on ESD, although according to Pavlova (2013) the similarities and differences between EE and ESD vary with regions and countries nowadays, as well as during the last 10, 20 or 30 years and, will continue to do so in the future. Pavlova (ibid) concludes that the EE and ESD should not only be evaluated one against the other, but on an independent, transformative educational reference system, that is, if there are qualities to lead to a transformative education.

ESD is the practice of learning how to manage to have global and local sustainable societies. Various approaches for ESD encourage people to understand the complexity and synergies among the issues that threaten the sustainability of our planet. In addition, through ESD students are encouraged to understand and evaluate their own values and those of the society they live in, within a sustainable framework.

Using science as a way of thinking and technology as the basic tools of solving problems gained a significant role in many sustainability subjects. The focus on solution paths through science and technological alternatives can overcome the obstacles to immediate action (Sarewitz et al, 2010).

In this paper, we will describe the method that was used to design and develop simulations that model everyday issues, mainly examples of heat transfer processes. These simulations are components of an educational framework designed to motivate students to think and act in a sustainable way. It is a Problem Oriented Project Based Learning (POPBL) environment, which enhances users to have an active role in the learning process, by using a variety of tools, such as LMS (Learning Management System) platforms, simulations, real time remote experiment and comprehend heat transfer mechanisms in various scales, such as understand the process of cooking, inquire the facts that affect the internal temperature of a house or the concept of low or zero energy houses.

2. Method

We designed, developed and embedded user-friendly simulations referring to everyday life phenomena of heat transfer mechanisms, such as conduction, convection, and radiation. These simulations were built using the program Energy2D (Xie, 2010). It is a Computer-aided engineering (CAE) system for teaching various topics of science and engineering through virtual experiments, offering the possibility of experimentation, variation, and designing an experiment.

Energy2D was created in 2010 by Charles Xie (2012), a physicist working at the Concord Consortium based in Concord, Massachusetts, USA. Xie invented a semi-Lagrangian McCormack method to approximately solve the Navier-Stokes equation. The performance of this solver is comparable to Jos Stam's unconditionally stable fluid solver, which cannot be used in open-source projects, because a large company claims it as part of a pending patent. Although the McCormack solver is not unconditionally stable, it is capable of simulating turbulent flows (higher Reynolds number) without introducing vorticity confinement (Xie, 2010). In the mathematical analysis below, we will describe heat flux and airflow equations used by Energy2D, to understand the basics of their solution.

3. Mathematical analysis of Enegry2D models

3.1. Studying heat flux

The simulations are models for representing heat flux through heterogeneous media and fluids. These models are based on solving heat and fluid equations using numerical methods. According to Xie (2012), assuming the phenomenon where there is a heat source and energy-heat diffuses through media – material and air in a three dimensional space, than based on energy consumption, the energy that has been convected (transferred through materials and fluids in general) equals the energy that has been advected (by macroscopic molecular motion) plus the energy, that has been conducted (heat transfer from energetic to less energetic particles due to an energy gradient). The following partial differential equation describes the above in the most sufficient way:

$$\rho c \left[\frac{\partial T}{\partial t} + \nabla \cdot (\mathbf{v}T) \right] = \nabla \cdot [k \nabla T] + q, \tag{1}$$

where, k is the thermal conductivity, c is the specific heat capacity, ρ is the density, $\mathbf{v}$ is the velocity field, and q is the internal heat generation which can be thought as an external force.

Equation (1) can be solved numerically using Finite Difference Time Domain (FDTD) methods. An implicit FDTD method can be used to achieve better numeric stability and using the relaxation method the equation can be solved. The boundary conditions that have been used are the Dirichlet and Neumann boundary conditions for partial differential equations. In the Dirichlet boundary condition, the temperature is fixed at the boundary and in the Neumann boundary condition, the flux is fixed at the boundary (Xie, 2010c).

3.2. Airflow and Navier-Stokes equations

Navier Stokes equations are the most common equation that are used to model fluid dynamics. They are named after Claude-Louis Navier and George Gabriel Stokes, and are used to describe the motion of viscous fluids. These equations are characterized as balanced equations. They are based on Newton's second law applied to fluid motion and on the assumption that the stress in the fluid is the sum of a diffusing viscous term, which is proportional to the gradient of velocity, and a pressure term, and are therefore describing viscous flow.

The Navier-Stokes equations found many applications in a wide range of scientific fields. In meteorology it has been a useful tool for weather-forecasting and prediction of storms (Shapiro, 1993). They have also been applied in the study of the ocean circulation systems and ocean currents (Marshall et al, 1997). The study of water flow in different kind of pipes is of great significance in the field of engineering (Ward-Smith, 1980). Aerodynamic design is a field where the Navier-Stokes equation is very useful and it has been used for aerodynamic shape design optimization (Nielsen & Anderson, 1999) as well as for the study of flying mechanisms in living creatures (Liu & Kawachi, 1998). An application in biological systems, is to use of the Navier-Stokes equation for the study of blood flow in the heart (Perkin, 1977) and vessels (Perktold & Rappitsch, 1995). The coupling between the Navier-Stokes equation and the Maxwell equation has been used to model and study magnetohydrodynamics (Gerbeau, 1998).

Studying the airflow as an incompressible Newtonian fluid and its contribution to the convective effect and to natural convection, it can be described through the following form of the Navier Stokes equation:

$$\frac{\partial \mathbf{v}}{\partial t} = a\nabla^2\mathbf{v} - (\mathbf{v}\nabla)\mathbf{v} - \nabla p + \mathbf{f}, \tag{2}$$

where $\mathbf{v}$ is the velocity vector, a is the kinematic viscosity, p is the pressure, and $\mathbf{f}$ is the body force such as gravity or buoyancy. The first term on the right-hand side describes the diffusion of momentum, and the second term describes advection. Equation (2) can be solved, according to Xie (2010d), by decomposing it in two steps, the diffusion and the advection step, studying it separately and still trying to satisfy the principle of conservation of mass in each step. Furthermore:

– the diffusion step

$$\frac{\partial \mathbf{v}_1}{\partial t} = \alpha\nabla^2\mathbf{v}_1 + \mathbf{f} \tag{3}$$

can be solved using a relaxation method as in the case of the heat equation above;

– the advection step

$$\frac{\partial \mathbf{v}_2}{\partial t} = -(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2 \tag{4}$$

can be solved using the MacCormack method.

Finally, each step should be imposed by the continuity condition $\nabla \cdot \mathbf{v} = 0$. Analyzing the above to a discretized form, based on the Helmholtz decomposition, and applying on it the gradient operator, it becomes the Poisson equation. By discretizing again, this equation can be solved using the relaxation method with the iteration above until convergence is reached.

3.3. Developing custom simulations

Our goal was to develop simulations that can model phenomena of the everyday life to which a user can relate to. That's why concepts are based on studying heat propagation through familiar materials and concepts. Energy 2D is a java based desktop app, which gives the opportunity of setting material and environment properties, enables the possibility to design objects easily and exports it as an applet, so that it can be embedded in websites. We created virtual experiments, where a user can study the temperature behavior of different materials and environments in a time range.

Below we will present some of the simulations. The simulations cover everyday phenomena like cooking, using a covered and an uncovered pot, and range to buildings, which is a much more complicated project, where students are encouraged to experiment with a house in insulation and a house without insulation. In the building all parameters can be changed, as the thickness of the walls, the size of the window, windows can be added or removed, the house can have a tile roof or not, the angle of the sun and the intensity of the sunshine, etc. In this way students can observe how all these influence the temperature in the building.

We start with the "Cooking experiment". In this concept, we designed a simplified kitchen, where there are two pots made of conductive materials. Students can observe the cooking process when one pot is covered with a lid and the other is not. Questions like "In which pot will the water boil quicker", or "In which case do we observe more loss of heat to the environment", etc can be addressed. In Fig. 1 we depict the initial state of the experiment. In both pots thermometers are placed in order to observe the change of the temperature in each pot. Students may also add thermometers to several points of the environment in order to check the temperature change of it.

Material properties: both, pots and cooking materials, are highly conductive and have the initial temperature of 20 degrees of Celsius. Thermal properties of the above are initiated as shown in Table 1.

Tab. 1: Thermal properties of the materials used in the cooking experiment.

Material	Termal conductivity [W/m°C]	Specific heat [J/kg°C]	Density [Kg/m^3]
Pot	1	1300	25
Heat source (constant temperature of 50°C	1	1300	25
Cooking material	1	1300	25

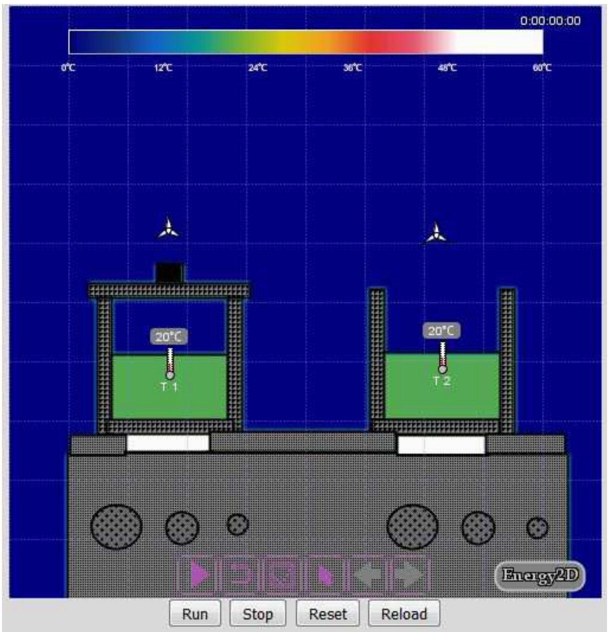

Fig. 1: The cooking experiment.

The air conductivity is to 0.024 W/m°C, specific heat to 1012 J/kg°C, density to 1.225 kg/m^3, kinematic viscosity to 0.12170293 m^2/s.

The model has a time step length of 0.1 s and thermometers are placed inside of each pot, so users can observe their behavior through time.

The second experiment we present is the "Zero energy house". The simulation below, is a component of an educational framework called "Learning about zero energy houses", intended to help students to understand heat transfer processes starting from the mentioned simplified concepts and use their results and conclusions to more complicated situations, like the thermal behavior of a zero energy house. The following two simulations were designed in order to model the passive thermal behavior of a house that uses the sun as a thermal source. Both simulations were "built" using building materials like bricks, wood, glass, etc. The main difference is that the one has an additional layer of high insulated material, as shown below in Fig. 2.

The properties of the materials we used in this simulation are presented in Table 2.

4. Results

By running these simulations, users collect temperature data or analyze directly the diagrams from indications of the thermometers that were placed on each model. Below there are some data about each model:

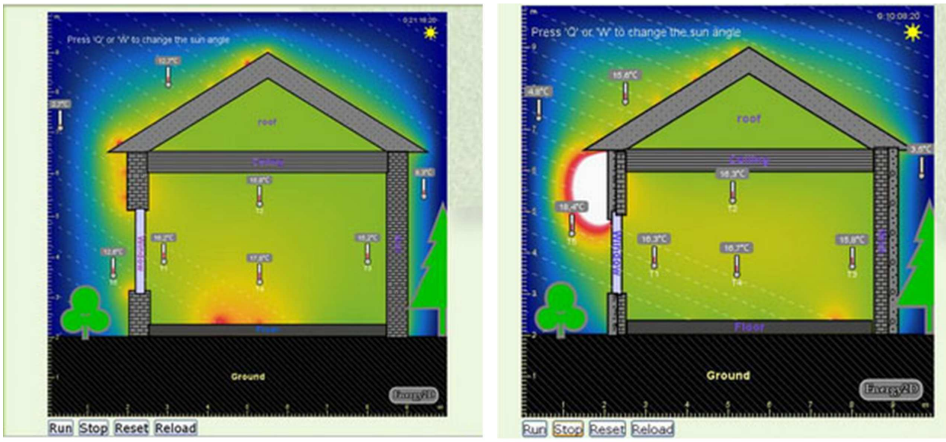

Fig. 2: House models with and without insulation.

Tab. 2: The thermal properties of the building materials used in the house models.

Hause models			
Material	Termal conductivity [W/m°C]	Specific heat [J/kg°C]	Density [kg/m^3]
roof	0.84	800	1900
walls	0.84	800	1900
walls (insulation)	0.035	1400	25
ceiling	1.13	1000	200
floor	1.13	1000	200
window	0.001	1300	25

4.1. The cooking experiment

Thermometers are placed inside each pot, in the same position and as the simulation runs, we can observe the change of the temperature in each pot. Thus we can watch in which case (pot) our food will be cooked quicker. By running the simulations we can choose if we want to see the simulation only, by reading the temperatures on the thermometers, or if we also want to have the temperatures of the thermometers, T_1 and T_2, depicted in graphs at the same time, as shown in Fig. 3, below. The coloration of the background depicts the temperature, starting with blue for the lowest temperature and white the highest. From that we can also infer the heat flow to the environment

We can also get the results in a table for further examination as presented in Fig. 4, where we get the readings of the thermometers at a specified time interval, in our case 10 seconds.

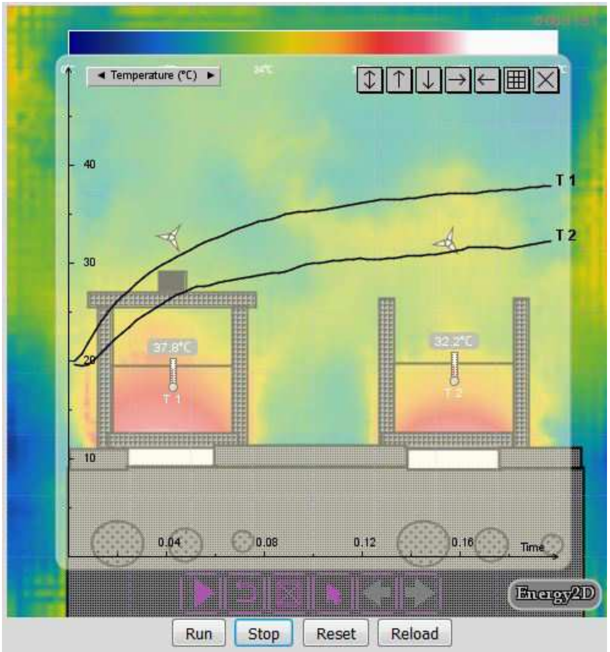

Fig. 3: Running the cooking experiment simulation.

The results from the above experiment show that in the covered pot, the temperature is better maintained and increased quicker than in the uncovered one. This conclusion can be used in cooking or warming up food in our everyday live, but also to stimulate discussions as for example by covering a hot beverage it can maintain its high temperature for a longer time. This finding can furthermore be used to motivate discussions about other similar issues or everyday phenomena. Such an important example is the ceiling in house's roof.

4.2. Low energy consumption houses

The following simulations, which are actually quite similar as described above, represent house models, with the aid of which we can study their thermal behavior. Thermometers are placed inside and outside the house model. More specifically, there are two thermometers outside the house – in the environment of the building – one in front and the other behind the house model. In addition, inside the house there are more thermometers placed in different positions, as for example, in front of the window, in the middle of the room, close to the ceiling, in the back of the room. The user is free to add and position in specific positions thermometers in order to observe the change of the temperature and the temperature distribution inside the building.

Thermometers (℃)

Time	T 1	T 2
10.0	19.909754	19.524267
20.0	20.671726	19.375015
30.0	21.866877	19.613718
40.0	23.037289	20.16835
50.0	24.092314	20.832628
60.0	25.02904	21.515287
70.0	25.856565	22.146595
80.0	26.587402	22.77878
90.0	27.249315	23.446594
100.0	27.852964	24.025494
110.0	28.41302	24.533615
120.0	28.938213	24.945604
130.0	29.416637	25.375366
140.0	29.840115	25.826603
150.0	30.217865	26.267832
160.0	30.583683	26.658121
170.0	30.963392	26.984478
180.0	31.34119	27.27751
190.0	31.7071	27.536985
200.0	32.018734	27.617125
210.0	32.28779	27.703138
220.0	32.564934	27.898457
230.0	32.826096	28.074203
240.0	33.128166	28.1571
250.0	33.443783	28.293001
260.0	33.702408	28.477537
270.0	33.936634	28.6158
280.0	34.132946	28.68475
290.0	34.307137	28.845104
300.0	34.475483	28.947039
310.0	34.684536	28.984266
320.0	34.884224	29.057644
330.0	35.037582	29.293842
340.0	35.143913	29.586239
350.0	35.19607	29.753027
360.0	35.260178	29.887403
370.0	35.347187	29.942175
380.0	35.473316	30.000135

Copy Data Close

Fig. 4: Data of the process can be exported in a table for further analysis.

Below, in Fig. 5, we present a screenshot of the simulation of a house model without insulation.

There are many scenarios we can follow by running these simulations. We can change the angle of the sun – in this way the sunrays can enter deeper into the room. We can change the size of the window (allowing more or less sunrays to enter into the room), or the orientation of the window (showing that it is not indifferent in which direction the window is placed). We can also change the thickness of the walls (changing thus the heat capacity of the building material). We can furthermore examine how small changes of the form of the house, as for example the addition of

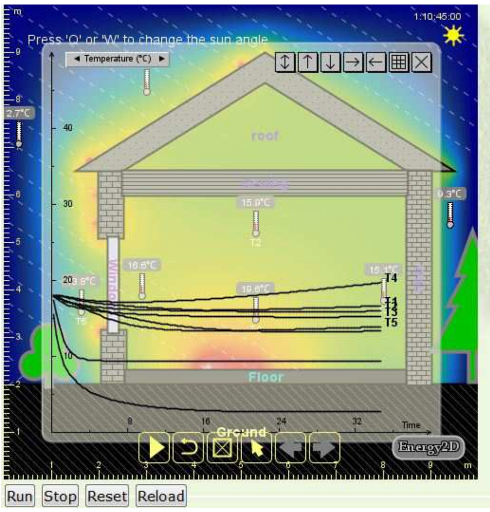

Fig. 5: Screenshots of the simulation of the house model without insulation.

Thermometers (℃)

Time	T5	T2	T3	T1	T4	Value	Value	Value
1000.0	17.969296	17.999113	17.846125	17.965223	17.999851	17.251934	17.956505	15.376294
2000.0	17.823189	17.98019	17.521858	17.81939	17.992691	15.598743	17.76987	12.642028
3000.0	17.652258	17.927658	17.252203	17.638994	17.96492	14.045138	17.561497	10.898712
4000.0	17.498365	17.852192	17.043486	17.46986	17.916546	12.814097	17.381086	9.703545
5000.0	17.360039	17.767067	16.877174	17.321806	17.855085	11.885862	17.231632	8.823807
6000.0	17.227692	17.680395	16.739758	17.193842	17.787365	11.199391	17.107094	8.141687
7000.0	17.09363	17.596247	16.622774	17.082771	17.7181	10.696369	17.000536	7.5922594
8000.0	16.953724	17.516418	16.520903	16.985462	17.650267	10.329572	16.90599	7.136882
9000.0	16.806784	17.441479	16.430683	16.899382	17.585697	10.062865	16.818802	6.7510777
10000.0	16.653399	17.371546	16.34978	16.822529	17.525412	9.869242	16.73556	6.418533
11000.0	16.495073	17.306364	16.276506	16.753393	17.46988	9.728744	16.653942	6.1279173
12000.0	16.333725	17.245604	16.209661	16.690786	17.419252	9.626779	16.572426	5.8710785
13000.0	16.171267	17.188843	16.148304	16.633778	17.37344	9.552724	16.490128	5.64198
14000.0	16.009445	17.135664	16.091722	16.581635	17.332247	9.4988575	16.406624	5.4360394
15000.0	15.849788	17.08568	16.039328	16.533796	17.295397	9.459564	16.321785	5.2496986
16000.0	15.693524	17.038553	15.990641	16.489754	17.262589	9.430792	16.235685	5.0801463
17000.0	15.541643	16.993956	15.945265	16.449114	17.233494	9.409646	16.148485	4.9251256
18000.0	15.39489	16.951584	15.902868	16.411533	17.207838	9.3940115	16.060482	4.782799
19000.0	15.253803	16.911194	15.863149	16.376732	17.185316	9.382362	15.971943	4.651648
20000.0	15.118757	16.87255	15.825871	16.344467	17.16566	9.373584	15.88317	4.5304065
21000.0	14.989966	16.835482	15.790809	16.314507	17.148655	9.366904	15.794473	4.418008
22000.0	14.867539	16.799803	15.757764	16.286676	17.134066	9.361744	15.706121	4.313544
23000.0	14.751496	16.765371	15.726582	16.260798	17.121714	9.357685	15.618374	4.216232
24000.0	14.641775	16.732082	15.697102	16.236742	17.111408	9.354421	15.531473	4.1254096
25000.0	14.538255	16.699825	15.669197	16.214355	17.103022	9.351768	15.445612	4.0404787
26000.0	14.440788	16.6685	15.642748	16.19354	17.096403	9.349545	15.360983	3.9609327
27000.0	14.349178	16.638056	15.617654	16.17417	17.09144	9.347634	15.277745	3.8863192
28000.0	14.263227	16.60841	15.593821	16.15617	17.088017	9.345965	15.196038	3.816238
29000.0	14.182708	16.579529	15.571157	16.139437	17.086046	9.34448	15.115976	3.750334
30000.0	14.107382	16.551363	15.549593	16.12391	17.085463	9.343123	15.037666	3.68829
31000.0	14.037022	16.52387	15.529051	16.109507	17.086166	9.3418665	14.96117	3.629819
32000.0	13.971388	16.497046	15.509484	16.096184	17.08811	9.3407	14.886566	3.5746672
33000.0	13.910251	16.470856	15.490826	16.08385	17.09122	9.339607	14.813897	3.5226042
34000.0	13.853378	16.445267	15.473006	16.07247	17.095451	9.3385515	14.743202	3.4734142
35000.0	13.800544	16.420292	15.455988	16.062002	17.100746	9.337546	14.674513	3.4269087
36000.0	13.751532	16.395899	15.439731	16.052372	17.107065	9.336564	14.607833	3.3829174
37000.0	13.706136	16.372065	15.424181	16.043554	17.11437	9.335621	14.543185	3.3412771

Copy Data Close

Fig. 6: Exported data from the simulation of the house without insulation.

a ceiling or a tiled roof has on its thermal behavior, or even what happens if we have both, a ceiling and a tiled roof. We can also change the position of the vegetation etc, thus influencing the amount of sunrays that reach the window and exterior building elements and study the thermal behavior of these house models. We can analyze the influence of these differences on the thermal behavior of the various house models by comparing the values of the thermometers that are placed indoors and outdoors, calculating and comparing their mean value or analyzing the diagram.

From the simulation of the above scenario as shown in Fig. 6, we can compare the indoor and outdoor temperature difference, but we can also observe a temperature dispersion inside the house, with places that are warmer or colder than other parts. This indicates that the house does not always offering a comfortable living environment, with parts that are much colder.

4.3. The insulated house

In order to study the influence of insulation to a building, we have to compare two similar buildings, one without and one with insulation.

Fig. 7: Simulation of the house model with insulation.

In this house model we can also test a number of different scenarios, in order to check their influence of the thermal behavior of the house. We can run the scenarios mentioned previously and compare the results to the ones with the house model without insulation material. We can furthermore study the difference of putting the insulation inside or outside the walls in the house model. The above results can be the starting point for several discussions and conclusions of the factors that affect the internal temperature of houses, and more particular of zero energy houses. Comparing the results of the two simulations which have the same conditions, we can observe the indoor-outdoor temperature difference in the two model houses and realize that insulated houses are warmer during cold weather than the ones without insulation. We can also observe the temperature stability that an insulated house can achieve, compared to the one without the insulation materials.

5. Conclusions

Simulations can be highly powerful tools in an active learning process. Students can inquire knowledge, check by experiments others or their own assumptions, reach to conclusions and form positive attitudes towards a sustainable way of living. Thermal simulations using Energy2d enable the construction of applications with everyday phenomena, which even if they are based in complex mathematical and physical systems, give students the opportunity to simplify the process and experiment with the various forms of heat transfer. The results from the educational framework of these simulations were used showing high interest and participation from the students, better understanding of the phenomena, and better collaboration among them. Feedback on the simulations and their use in the educational process were very positive from both, the teachers and the students. Teachers suggested that each simulation separately could be used as a standalone application and develop separate projects of each one. This framework is still in progress and part of a research project about the effectiveness of polymorphic distant learning environments based on Education for Sustainable Development.

References

[1] S. A. Barab, M. A. Evans, and E.-O. Baek, *Activity theory as a lens for characterizing the participatory unit*, in: Handbook of Research for Educational Communication and Technology (2nd ed.), D. H. Jonassen (ed.), Mahwah, Lawrence Erlbaum Associates, NJ 2004, pp. 199–214.

[2] D. Bolscho and K. Hauenschild, *From environmental education to Education for Sustainable Development in Germany*, Environmental Education Research **12**, no. 1 (2006), 7–18.

[3] M. Fullan and A. Ballew, *Leading in a Culture of Change: Personal Action Guide and Workbook*, Jossey-Bass, San Franscisco 2001.

[4] J. F. Gerbeau, *A stabilized finite element method for the incompressible magnetohydrodynamic equations*, Numerische Mathematik **87**, no. 1 (2000), 83–111.

[5] H. R. Hungerford, R. B. Peyton, and R. J. Wilke, *Goals for curriculum development in environmental education*, Journal of Environmental Education **2**, no. 3 (1980), 42–47.

[6] A. Jameson, N. A. Pierce, and L. Martinelli, *Optimum aerodynamic design using the Navier Stokes Equations*, AIAA 35th Aerospace Sciences Meeting & Exhibit (1997), doi: 10.2514/6.1997-101.

[7] H. Liu, H. and K. Kawachi, *A numerical study of insect flight*, Journal of Computational Physics **146**, no. 1 (1998), 124–156.

[8] J. Marshall, A. Adcroft, C. Hill, L. Perelman, and C. Heisey, *A finite-volume, incompressible Navier Stokes model for studies of the ocean on parallel computers*, Journal of Geophysical Research **102** (C3), (1997), 5753–5766.

[9] R. McKeown and C. Hopkins, *EE$\neq$ESD: Defusing the worry*, Environmental Educational Research **9**, no. 1 (2003), 117–128.

[10] E. J. Nielsen and W. K. Anderson, *Aerodynamic design optimization on unstructured meshes using the Navier-Stokes equations*, AIAA Journal **37**, no. 11 (1999), 1411–1419.

[11] T. O'Riordan, *Environmentalism*, Pion Ltd, London 1976.

[12] M. Pavlova, *Towards using transformative education as a benchmark for clarifying differences and similarities between Environmental Education and Education for Sustainable Development*, Environmental Education Research **19**, no. 5 (2013), 656–672.

[13] C. S. Perkin, *Numerical analysis of blood flow in the heart*, Journal of Computational Physics **25**, no. 3 (1977), 220-252.

[14] K. Perktold and G. Rappitsch, *Computer simulation of local blood flow and vessel mechanics in a compliant artery bifurcation model*, Journal of Biomechanics **28**, no. 7 (1995), 845–856.

[15] A. Reid, *Discussing the Possibility of Education for Sustainable Development*, Environmental Education Research **8**, no. 1 (2002), 73–79.

[16] D. Sarewitz, D. Kriebel, R. Clapp, C. Crumbley, P. Hoppin, M. Jakobs, and J. Tickner, *The Sustainable Solutions Agenda*, Consortium for Science, Policy and Outcomes, Arizona State University, and Lowell Center for Sustainable Production, University of Massachusetts, Lowell 2010.

[17] W. A. H. Scott, *ESD: What Sort of Decade? What Sort of Learning? Keynote Address at the UK Launch of the UNESCO Decade for ESD*, Center for Research in Education and the Environment, University of Bath, Bath 2005.

[18] W. A. H. Scott and S. Gough, *Sustainable Development and Learning: Framing the Issues*, Routledge Falmer, London 2003.

[19] A. Shapiro, *The use of an exact solution of the Navier-Stokes equations in a validation test of a three-dimensional nonhydrostatic numerical model*, Monthly Weather Review **121**, no. 8 (1993), 2420–2425.

[20] UNESCO (ed.), *Strategie Internationale d'Action en Matière d'Éducation et de Formation Relatives à l'Environnement pour les Années 1990*, U.N.E.S.C.O.-U.N.E.P. Congrés, Environmental education and training, (MOCKBA 1987), Nairobi-Paris 1988.

[21] UNESCO, *Education, Public Awareness and Training for Sustainablity: Input to the Report of the Secretary General to the Second Preparatory Session for the World Summit on Sustainable Development*, Unesco, Paris 2002.

[22] UNESCO, *United Nations Decade of Education for Sustainable Development 2005–2014. Draft Implementation Scheme*, October 2004; http://portal.unesco.org/ education/en/file download.php/03f375b07798a2a55dcdc39db7aa8211Final+IIS.pdf.

[23] A. J. Ward-Smith, *Internal Fluid Flow. The Fluid Dynamics of Flow in Pipes and Ducts*, Clarendon Press.

[24] World Commission on Environment and Development (WCED), Our common future, Oxford 1987, Oxford University Press.

[25] http://web.mit.edu/rsi/www/2005/misc/minipaper/papers/Hake.pdf

[26] C. Xie, *Interactive heat transfer simulations for everyone*, The Physics Teacher **50**, no. 4 (2012), 237–240.

[27] C. Xie, *Energy2D. Interactive heat transfer simulations for everyone*, Retrieved from http://energy.concord.org/energy2d/; 2010a.

[28] C. Xie, *Energy2D. Bringing powerful scientific and engineering simulations to your classroom*, Retrieved on 12 February 2013 from http://energy.concord.org/energy2d/about.html; 2010b.

[29] C. Xie, *Solving the heat equation*, Retrieved on 12 February 2013 from http://energy.concord.org/energy2d/papers/heat_solver.pdf; 2010c.

[30] C. Xie, *Solving the fluid equations*, Retrieved on 12 February 2013 from http://energy.concord.org/energy2d/fluid_solver.pdf; 2010d.

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Presented by Marek Moneta at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on January 27, 2016

PROCESY FIZYCZNE TRANSFERU ENERGII ZWIĄZANE Z CODZINNYM DOŚWIADCZENIAMI ŻYCIOWYMI

S t r e s z c z e n i e

W artykule opisujemy rozmaite podejścia do fizycznych procesów transferu energii, które mogą być wykorzystane w procesie nauczania z tego zakresu, zogniskowanym na meritum tematu. Dla osiągnięcia tego, zastał zastosowany polimorficzny zakres edukacyjny, co zawiera platformę nawiązującą do rozmaitych narzędzi edukacyjnych jak symulacje, rzeczywiste doświadczenia potwierdzające, prezentacje itd., które są w użyciu nauczyciela lub innej zainteresowanej osoby. Symulacje mogą być użyteczne przy modelowaniu zjawisk fizycznych w różnych skalach w tym samym czasie i mogą pomóc użytkownikom, by lepiej zrozumieć teoretyczną strukturę zjawiska wraz z zastosowaniem w realnym życiu.

Słowa kluczowe: transfer energii, równania Naviera-Stokesa, modelowanie a symulacja

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Contribution to the jubilee volume, dedicated
to Professors J. Ławrynowicz and L. Wojtczak

Jacek Hejduk, Władysław Wilczyński, and Wojciech Wojdowski

ON SEMI-REGULARIZATION OF THE DENSITY-TYPE TOPOLOGIES

Summary

We show that for several density topologies the semi-regularization of the topology is the coarsest topology for which the functions approximately continuous with respect to the topology are continuous. We show various methods of proof of this property and ask for characterization of density topologies with this property.

Keywords and phrases: density point, density topology, semiregularization

1. Introduction

Let (X, τ) be a topological space. By (X, τ_{sem}) we denote the semi-regularization of (X, τ) i.e. X with the topology generated by regular open subsets of X.

By (X, τ_{ini}) we denote the initial topology i.e. X with the topology induced by the family of τ-continuous functions from X to $\mathbb{R}$. The topology τ_{ini} is the coarsest topology for which the τ-continuous functions are continuous (see [AP], p. 51).

Let us recall that a space (X, τ) is completely regular if and only if τ is the coarsest topology for which the τ-continuous functions are continuous (see [AP], p. 51).

Recall that a space (X, τ) is completely regular if and only if $\tau = \tau_{\text{ini}}$.

The aim of this paper is to discuss more thoroughly some properties of the continuity of real functions with respect to several density topologies and to show that the semi-regularization of some of them is the coarsest topology for which the continuous functions with respect to the topology are continuous. At the end of the paper we pose a problem to find a characterisation of these density topologies for which the semi-regularization is the initial topology.

We shall prove that in case of several density topologies a stronger version of the theorem cited below is valid.

Proposition 1. (*Proposition* 1.14 *in* [AM]) *Let* (X, τ) *be a topological space,* (X, τ_{sem}) *the semi-regularization of* (X, τ) *i.e.* X *with the topology generated by regular open subsets of* X. *Let* Y *be a regular space. Then functions* $f : (X, \tau) \to Y$ *and* $f : (X, \tau_{\mathrm{sem}}) \to Y$ *are continuous simultaneously.*

It is also worth mentioning that (X, τ) is a topological Hausdorff space if and only if (X, τ_{sem}) is Hausdorff (compare [MR]).

We start with some overview of the notion of the density topology.

Let S be a σ-algebra of subsets of the real line $\mathbb{R}$, and $I \subset S$ a proper σ-ideal. In measure case we shall consider $S = \mathcal{M}$ the family of Lebesgue measurable sets and $I = \mathcal{N}$ the family of null sets. In category case we shall consider $S = \mathcal{B}$ the sets with the Baire property and $I = \mathcal{I}$ the first category sets. With λ we denote the Lebesgue measure on the real line. Further,

$$n \cdot A = \{nx : x \in A\}, \quad A - x_0 = \{x - x_0 : x \in A\}$$

and $A \sim B$ means that $A \triangle B \in I$. We assume that both S and I are closed under translations $A - x_0 = \{x - x_0 : x \in A\}$ and dilations $n \cdot A = \{nx : x \in A\}$.

Recall that the point $x \in \mathbb{R}$ is a Lebesgue density point of a measurable set A, if and only if

$$(*) \qquad \lim_{h \to 0} \frac{\lambda\left(A \cap [x-h, x+h]\right)}{2h} = 1$$

The notion of density point has been studied and developed extensively since the notion of density topology $\mathcal{T}_d$ was introduced by Haupt and Pauc in 1952 [HP]. It is interesting that the related notion of approximate continuity, as defined by Denjoy in 1915 [D], had been known far earlier and utilized in the study of the theory of integration. The properties of the density topology were discovered gradually by Goffman and Waterman [GW], Goffman, Neugebauer and Nishiura [GNN] and Tall [T]. The state of the art of the theory at the end of 70-ties was presented by J. C. Oxtoby in his excellent book [O]. However, in 1981 the definition of the density point was reformulated without the use of the notion of measure (see [W1]).

The condition $(*)$ in the definition is equivalent to any of the following:

$$\lim_{n \to \infty} \frac{\lambda\left(A \cap \left[x - \frac{1}{n}, x + \frac{1}{n}\right]\right)}{\frac{2}{n}} = 1$$

or

$$\lim_{n \to \infty} \lambda\left(n \cdot (A - x) \cap [-1, 1]\right) = 2$$

or

$$\left\{\chi_{(n \cdot (A - x)) \cap [-1,1]}\right\}_{n \in \mathbb{N}} \text{ converges in measure to } \chi_{[-1,1]}.$$

With the use of the Riesz theorem an equivalent definition was given, by Wilczyński in 1981, in terms of convergence almost everywhere of characteristic functions of dilations of the set A.

Definition 1. (Wilczyński 1981). *We say that x is a density point of a measurable set $A \in \mathcal{M}$ if for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero there is a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that the sequence*

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$$

of characteristic functions converges $\mathcal{N}$-almost everywhere (a.e.) on $[-1,1]$ to $\chi_{[-1,1]}$.

The reformulated definition presented the opportunity for the study of more subtle properties of the notion of the density point and density topology and its various modifications. It allowed to introduce a category analogue of the notion (see [PWW1], [PWW2], [CLO]).

Definition 2. (cf. [PWW1]) *We say that x is an $\mathcal{I}$-density point of a Baire set $A \in \mathcal{B}$ if for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there is a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that the sequence*

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$$

of characteristic functions converges $\mathcal{I}$-almost everywhere on $[-1,1]$ to $\chi_{[-1,1]}$ (which means except on a set belonging to $\mathcal{I}$).

The notion of the $\mathcal{I}$-density point leads in a natural way to the definition of the $\mathcal{I}$-density topology which appeared to have most the same topological properties of the Lebesgue density topology. This study extended thus the Oxtoby's investigation of similarities and dissimilarities of the properties of the families $(\mathcal{M},\mathcal{N})$ and $(\mathcal{B},\mathcal{I})$.

In view of the above results the following definition in more general settings where (S,I) is an arbitrary pair of S, σ-algebra of subsets of the real line $\mathbb{R}$ and $I \subset S$ a proper σ-ideal, is quite natural (see [PWW1]).

Definition 3. *A point $x \in \mathbb{R}$ is an I-density point of a set $A \in S$, if for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there is a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that the sequence*

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$$

of characteristic functions converges I-almost everywhere on $[-1,1]$ to $\chi_{[-1,1]}$ (which means except on a set belonging to I).

The set of all I-density points of $A \in S$ will be denoted by $\Phi_I(A)$.

2. Properties

We shall present now some properties of regular open sets in a general case.

Let $\mathbf{B} \subset 2^{\mathbb{R}}$ be a σ-algebra and $\mathbf{I} \subset \mathbf{B}$ a σ-ideal containing singletons and such that $\bigcup \mathbf{I} = \mathbb{R}$.

We assume that for every $A \subset \mathbb{R}$ there is a $\mathbf{B}$-kernel of A i.e. a subset $B \in \mathbf{B}$ of A such that all $\mathbf{B}$-subsets of $A \setminus B$ are sets from $\mathbf{I}$ (compare [M]).

Let $\Phi : \mathbf{B} \to 2^{\mathbb{R}}$ have the following properties:

(1) for each $A \in \mathbf{B}$, $\Phi(A) \sim A$,

(2) for each $A, B \in \mathbf{B}$, if $A \sim B$, then $\Phi(A) = \Phi(B)$,

(3) $\Phi(\emptyset) = \emptyset$, $\Phi(\mathbb{R}) = \mathbb{R}$,

(4) for each $A, B \in \mathbf{B}$, $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.

The family $\mathcal{T} = \{A \in \mathbf{B} : A \subset \Phi(A)\} = \{\Phi(A) \setminus P,\ A \in \mathbf{B},\ P \in \mathbf{I}\}$ is a topology. We call it a density-type topology generated by the operator Φ which is called lower density operator on $(\mathbf{B}, \mathbf{I})$.

Remark 1. *Observe that*

a) *we have* $\Phi : \mathbf{B} \to \mathbf{B}$ *in view of* (1).

b) Φ *is monotonic since if* $A \subset B$ *we have* $\Phi(A) = \Phi(A \cap B) = \Phi(A) \cap \Phi(B) \subset \Phi(B)$,

c) Φ *is idempotent by* (1) *and* (2).

Theorem 1. *For an arbitrary set* $A \subset \mathbb{R}$

$$Int_{\mathcal{T}}(A) = A \cap \Phi(B),$$

where B *is an* $\mathbf{B}$*-kernel of* A.

Proof. We can follow here the proof of Theorem 2.5. from [W2] (compare [Os]). Let $x \in Int_{\mathcal{T}}(A)$. Then there exists a set $U \in \mathcal{T}$ such that $x \in U$ and $U \subset A$. So $x \in \Phi(U)$. Since $U \setminus B \subset A \setminus B$ and $U \setminus B \in \mathbf{B}$ we have $U \setminus B \in \mathbf{I}$. Hence $\Phi(U) = \Phi(U \cap B) \subset \Phi(B)$ and $x \in A \cap \Phi(B)$.

Suppose now that $x \in A \cap \Phi(B)$ and we can find $P \in \mathbf{I}$ such that $x \in P \subset A$. Then $B \cup P \subset A$ and $B \cup P \in \mathbf{B}$, so $\Phi(B \cup P) = \Phi(B)$. Hence $x \in (B \cup P) \cap \Phi(B \cup P)$. Since Φ is idempotent and $\Phi((B \cup P) \cap \Phi(B \cup P)) = \Phi(B \cup P) \cap \Phi(\Phi(B \cup P)) = \Phi(B \cup P)$, the set $(B \cup P) \cap \Phi(B \cup P)$ is $\mathcal{T}$-open. So there exists a $\mathcal{T}$-open set including x and included in A. Hence $x \in Int(A)$. □

Theorem 2. *A set* $A \in \mathbf{B}$ *is* $\mathcal{T}$*-regular open if and only if* $A = \Phi(D)$ *for some* $D \in \mathbf{B}$.

Proof. Let A be $\mathcal{T}$-regular open, i.e., $A = Int_{\mathcal{T}}(Cl_{\mathcal{T}}(A))$. Then $A = Cl_{\mathcal{T}}(A) \cap \Phi(Cl_{\mathcal{T}}(A)) = Cl_{\mathcal{T}}(A) \cap \Phi(A) = \Phi(A)$. The first equality follows from the above theorem, the second from the fact that $Cl_{\mathcal{T}}(A) \sim A$ (because $Cl_{\mathcal{T}}(A) = \mathbb{R} \setminus Int_{\mathcal{T}}(\mathbb{R} \setminus A)$ and $Int_{\mathcal{T}}(\mathbb{R} \setminus A) = (\mathbb{R} \setminus A) \cap \Phi(\mathbb{R} \setminus A) \sim (\mathbb{R} \setminus A\ )$ and from (1), the third from the inclusion $\Phi(A) \subset Cl_{\mathcal{T}}(A)$. To justify the last inclusion we can write $Cl_{\mathcal{T}}(A) = \mathbb{R} \setminus Int_{\mathcal{T}}(\mathbb{R} \setminus A) = \mathbb{R} \setminus ((\mathbb{R} \setminus A) \cap \Phi(\mathbb{R} \setminus A)) = A \cup (\mathbb{R} \setminus \Phi(\mathbb{R} \setminus A))$.

Since

$$\Phi(A) \cap \Phi(\mathbb{R} \setminus A) = \emptyset,\ \Phi(A) \subset \mathbb{R} \setminus \Phi(\mathbb{R} \setminus A) \subset A \cup (\mathbb{R} \setminus \Phi(\mathbb{R} \setminus A)) = Cl_{\mathcal{T}}(A).$$

Suppose now that $A = \Phi(A)$. We have

$$Int_{\mathcal{T}}(Cl_{\mathcal{T}}(A)) = Cl_{\mathcal{T}}(A) \cap \Phi(Cl_{\mathcal{T}}(A)) = Cl_{\mathcal{T}}(A) \cap \Phi(A) = \Phi(A)) = A.$$

The verification of the first, second and the third equalities is exactly as above, the fourth is simply our assumption. □

From now **B** is the family of Baire sets $\mathcal{B}$ and **I** is the family of sets of the first category $\mathcal{I}$. We assume now additionally that $A \subset \Phi(A)$ (equivalently $A \in \mathcal{T}$), for every A open in $\mathcal{T}_n$ – the natural topology.

Recall that any set A having the Baire property can be presented in a unique way in the form $G(A) \triangle P$ where $G(A)$ is regular open and P is of the first category.

Remark 2. *As a consequence of* (2) *for every* $A \in \mathcal{B}$ *we have* $\Phi(A) = \Phi(G(A))$, *since* $G(A) \sim A$.

Theorem 3. *If a function* $f : \mathbb{R} \to \mathbb{R}$ *is* $\mathcal{T}$ *continuous, then* $f^{-1}((a,b))$ *is a countable union of* $\mathcal{T}$*-regular open sets, for every* $(a,b) \subset \mathbb{R}$.

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{T}$ continuous function. Let $(a,b) \subset \mathbb{R}$. We have $f^{-1}((a,b)) = \Phi(A) \setminus P$ for some $A \in \mathcal{B}$ and $P \in \mathcal{I}$, and we can assume $P \subset \Phi(G(A))$.

Let $x \in \Phi(A)$. We shall show that $f(x) \in [a,b]$. Really, as f is $\mathcal{T}$ continuous, for every $\epsilon > 0$ the set $f^{-1}((f(x) - \epsilon, f(x) + \epsilon)) \in \mathcal{T}$ and contains x. Since

$$(\{x\} \cup (\Phi(A) \setminus P)) \sim \Phi(A)$$

we have

$$(\{x\} \cup (\Phi(A) \setminus P)) \subset \Phi(\{x\} \cup (\Phi(A) \setminus P))$$

and thus also $x \in \{x\} \cup (\Phi(A) \setminus P) \in \mathcal{T}$. Thus

$$f^{-1}((a,b)) \cap f^{-1}((f(x) - \epsilon, f(x) + \epsilon)) \neq \emptyset$$

and belongs to $\mathcal{T}$.

Hence for every ϵ the set (a,b) intersects with $(f(x) - \epsilon, f(x) + \epsilon)$ and therefore $f(x) \in [a,b]$ ($x \in f^{-1}([a,b])$). We have $f^{-1}((a,b)) \subset \Phi(A) \subset f^{-1}([a,b])$.

We can repeat the above considerations for every set of the form $\left(a + \frac{1}{n}, b - \frac{1}{n}\right)$, where $n \in \mathbb{N}$. We may write

$$f^{-1}\left(a + \frac{1}{n}, b - \frac{1}{n}\right) = \Phi(A_n) \setminus P_n,$$

where $A_n \in \mathcal{B}$ and $P_n \in \mathcal{I}$, $n \in \mathbb{N}$ and we can assume $P_n \subset \Phi(A_n)$, $n \in \mathbb{N}$. We have

$$f^{-1}((a,b)) = \bigcup_{n=1}^{\infty} f^{-1}((a+\frac{1}{n}, b-\frac{1}{n})) \subset \bigcup_{n=1}^{\infty} \Phi(A_n)$$

$$\subset \bigcup_{n=1}^{\infty} f^{-1}([a+\frac{1}{n}, b-\frac{1}{n}]) = f^{-1}((a,b)).$$

Thus

$$f^{-1}((a,b)) = \bigcup_{n=1}^{\infty} \Phi(A_n).$$

We presented $f^{-1}((a,b))$ as a countable union of $\mathcal{T}$-regular open sets. □

Remark 3. *From Proposition* 1.14 *in* [AM] *we have that the set* $f^{-1}((a,b))$ *is open in the semiregularization of* $\mathcal{T}$*. Above we have shown that the set* $f^{-1}((a,b))$ *is a sum of only countably many* $\mathcal{T}$*-regular open sets.*

In [OM] O'Malley proved that the family of sets fulfilling condition $A \in \mathcal{T}_d$, where $\mathcal{T}_d$ is density topology, and $\lambda(A) = \lambda(Int(A))$ (called almost open) forms a topology (called *a.e.*-topology). He proved that *a.e.*-topology is completely regular but not normal.

The *a.e.*-open sets can be characterized as sets of the form $G \cup S$ where G is open in the natural topology and $S \subset \Phi_d(G)$, where Φ_d is the Lebesgue density operator. Clearly, S can be nowhere dense here.

By analogy, we define $\mathcal{T}$-*a.e.*-topology as a family of sets of the form $G \cup S$ where G is open in the natural topology and $S \subset \Phi(G)$.

Theorem 4. *If a function* $f : \mathbb{R} \to \mathbb{R}$ *is* $\mathcal{T}$*-approximately continuous, then it is continuous in* $\mathcal{T}$*-a.e. topology.*

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{T}$-approximately continuous function. Let $(a,b) \subset \mathbb{R}$. From the proof of the Theorem 3 we have

$$f^{-1}((a,b) = \bigcup_{n=1}^{\infty} \Phi(A_n),$$

where $A_n \in \mathcal{B}$. By (2) we have $\Phi(A_n) = \Phi(G(A_n))$, where $G(A_n)$ is the regular open set such that $A_n = G(A_n) \triangle P_n$, $n \in \mathbb{N}$. As assumed, since $G(A_n) \in \mathcal{T}_n$, we have $G(A_n) \subset \Phi(G(A_n))$ and thus

$$f^{-1}((a,b) = \bigcup_{n=1}^{\infty} \Phi(A_n) = \bigcup_{n=1}^{\infty} \Phi(G(A_n)) \supset \bigcup_{n=1}^{\infty} G(A_n).$$

We put $G = \bigcup_{n=1}^{\infty} G(A_n)$, and $S = f^{-1}((a,b) \setminus \bigcup_{n=1}^{\infty} G(A_n)$.

Clearly $\Phi(G(A_n)) \subset \Phi(\bigcup_{n=1}^{\infty} G(A_n))$ by monotonicity of Φ.

Then

$$\bigcup_{n=1}^{\infty} \Phi\left(G(A_n)\right) \subset \Phi(\bigcup_{n=1}^{\infty} G\left(A_n\right))$$

and by monotonicity and idempotency of Φ

$$\Phi(\bigcup_{n=1}^{\infty} \Phi\left(G(A_n)\right)) \subset \Phi(\Phi(\bigcup_{n=1}^{\infty} G\left(A_n\right))) = \Phi(\bigcup_{n=1}^{\infty} G\left(A_n\right))$$

and finally

$$\begin{aligned} S &= f^{-1}((a,b) \setminus \bigcup_{n=1}^{\infty} G\left(A_n\right) \\ &= \Phi(\bigcup_{n=1}^{\infty} \Phi\left(G(A_n)\right) \setminus \bigcup_{n=1}^{\infty} G\left(A_n\right)) \subset \Phi(\bigcup_{n=1}^{\infty} G\left(A_n\right)) = \Phi\left(G\right). \end{aligned}$$

□

3. Methods and examples

We shall present several examples of density-type topologies for which semi-regularization is the initial topology and show various methods of proof.

1. Lebesgue density topology

It is a consequence of the fact that the density topology is completely regular.

2. I-density topology

$\mathcal{I}$-density point from Definition 3 leads to the lower density operator $\Phi_{\mathcal{I}}$ which generates the $\mathcal{I}$-density topology $\mathcal{T}_{\mathcal{I}} = \{A \in \mathcal{B} : A \subset \Phi_{\mathcal{I}}\left(A\right)\}$.

We denote the set of all $\mathcal{I}$-density points by $\Phi_{\mathcal{I}}\left(A\right)$ and the induced topology $\{A \in \mathcal{B} : A \subset \Phi_{\mathcal{I}}\left(A\right)\}$ by $\mathcal{T}_{\mathcal{I}}$.

We may use now the Theorem 2 (compare also [PWW1], [PWW2] and [CLO]).

Let $x \in B$ where B is open in $sem\mathcal{T}_{\mathcal{I}}$ a semi-regularization of $\mathcal{T}_{\mathcal{I}}$. By definition of semi-regularization there is a $\mathcal{T}_{\mathcal{I}}$-regular open set $A \subset B$ such that $x \in A$. Assume for simplicity that $x = 0$. We shall define a $sem\mathcal{T}_{\mathcal{I}}$-continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f\left(0\right) = 1$ and $A^c \subset f^{-1}\left(0\right)$. By above $A = \Phi_I\left(D\right)$ for some $D \in \mathcal{B}$. We have $\Phi_{\mathcal{I}}\left(D\right) = \Phi_{\mathcal{I}}\left(H\right)$, where $H = G\left(D\right)$ regular open set, and $x \in A = \Phi_{\mathcal{I}}\left(D\right) = \Phi_{\mathcal{I}}\left(H\right)$ i.e.:

For any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that the sequence

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(H-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$$

of characteristic functions converges to $\chi_{[-1,1]}$ $\mathcal{I}$-almost everywhere on $[-1,1]$.

By Lemma 2.2.4 in [CLO] there exists an interval set $E \subset H$, consisting of closed intervals such that 0 is an $\mathcal{I}$-density point of E.

Lemma 1. (*Lemma* 2.2.4 *in* [CLO]) *Let H be a regular open set. Then, the following are equivalent:*

(i) 0 *is an $\mathcal{I}$-density point of H;*

(ii) there exists an interval set $E \subset H$, consisting of closed intervals such that 0 *is an $\mathcal{I}$-density point of E.*

Define

$$f(x) = \begin{cases} 1 & x = 0 \\ \frac{dist(x,H^c)}{dist(x,H^c)+dist(x,E)} & x \neq 0 \end{cases}$$

It is easy to see that $f(0) = 1$, $A^c \subset H^c \subset f^{-1}(0)$ and that f is continuous on $\mathbb{R} \setminus \{0\}$. Moreover, f is $sem\mathcal{T}_{\mathcal{I}}$-continuous at 0 as $E \subset f^{-1}(1)$.

3. $\mathcal{T}_{\Psi c}$-topology

Let $\mathcal{C}$ denote a class of continuous increasing functions $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t \to 0^+} \Psi(t) = 0$.

Definition 4. *Given $\Psi \in \mathcal{C}$, a point $x \in \mathbb{R}$ is a Ψc-density point of a set $E \in \mathcal{B}$, if and only if*

$$\lim_{h \to 0^+} \frac{\lambda\left(G(E)' \cap [x-h, x+h]\right)}{2h \cdot \Psi(2h)} = 0.$$

It means that x is a Ψc-density point of $E \in \mathcal{B}$ if x is a Ψ-density point of $G(E)$.(see [WT])

The operator $\Phi_{\Psi c} : \mathcal{B} \to 2^{\mathbb{R}}$ was defined by

$\Phi_{\Psi c}(A) = \{x \in \mathbb{R} : x \text{ is a } \Psi c - \text{density point of } A\}$. Thus we have $\Phi_{\Psi c}(A) = \Phi_{\Psi}(G(A))$. It was proved that for each $\Psi \in C$, the mapping $\Phi_{\Psi c} : \mathcal{B} \to 2^{\mathbb{R}}$ is the lower density operator on $(\mathcal{B}, \mathcal{I})$.

Consequently, the $\mathcal{T}_{\Psi c}$-topology was defined as a family

$$\mathcal{T}_{\Psi c} = \{A \in \mathcal{B} : A \subset \Phi_{\Psi c}(A)\} = \{\Phi_{\Psi c}(A) \setminus P : \ A \in \mathcal{B},\ P \in \mathcal{I}\}$$

and shown in [WIWO] that for each $\Psi \in \mathcal{C}$ the family $\mathcal{T}_{\Psi c}$ is stronger than the natural topology and weaker than the $\mathcal{T}_c$-density topology and is not regular.

When we put Ψ to be constant and equal 1 in Definition 1, we obtain the definition of the c-density point leading to the lower density operator Φ_c which generates the $\mathcal{T}_c$-density topology [Wo1].

Let $x \in B$ where B is open in $sem\mathcal{T}_{\Psi c}$ a semi-regularization of $\mathcal{T}_{\Psi c}$. By definition of semi-regularization there is a $\mathcal{T}_{\Psi c}$-regular open set $A \subset B$ such that $x \in A$. Assume $x = 0$. We shall define a $sem\mathcal{T}_{\Psi c}$-continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$ and $A^c \subset f^{-1}(0)$. By above $A = \Phi_{\Psi c}(D)$ for some $D \in \mathcal{B}$. We have

$$\Phi_{\Psi c}(D) = \Phi_{\Psi}(G(D)), \quad G(D) \subset \Phi_{\Psi c}(D) = A$$

and

$$x \in \Phi_{\Psi c}(D) = \Phi_{\Psi}(G(D))$$

i.e.

$$\lim_{h\to 0^+}\frac{\lambda\left(G\left(D\right)'\cap\left[-h,h\right]\right)}{2h\cdot\Psi\left(2h\right)}=0.$$

Since $G\left(D\right)$ is regular open in every interval $\left(\frac{1}{n+1},\frac{1}{n}\right)$ we can find $\{I_k^n\}_{k=1}^{k_n}$ a finite number of pairwise disjoint open intervals such that

$$I_k^n\subset G\left(D\right)\cap\left(\frac{1}{n+1},\frac{1}{n}\right)$$

and closed intervals $\{J_k^n\}_{k=1}^{k_n}$, $J_k^n\subset I_k^n$ such that

$$\frac{\lambda\left(J_n'\cap\left(\frac{1}{n+1},\frac{1}{n}\right)\right)}{\lambda\left(G\left(D\right)'\cap\left(\frac{1}{n+1},\frac{1}{n}\right)\right)}<1+\frac{1}{n},$$

where $J_n=\bigcup\limits_{k=1}^{k_n}J_k^n$.

The set $E=\bigcup\limits_{n=1}^{\infty}J_n$ forms a closed interval set and since clearly

$$\lim_{h\to 0^+}\frac{\lambda\left(E'\cap\left[-h,h\right]\right)}{2h\cdot\Psi\left(2h\right)}=0$$

we have $x\in\Phi_\Psi\left(E\right)$. We also define an open interval set

$$V=\bigcup_{n=1}^{\infty}I_n,\quad\text{where}\quad I_n=\bigcup_{k=1}^{k_n}I_k^n.$$

Define

$$f\left(x\right)=\begin{cases}1 & x=0,\\ \frac{dist(x,V^c)}{dist(x,V^c)+dist(x,E)} & x\neq 0\end{cases}.$$

It is easy to see that $f\left(0\right)=1$, $B^c\subset V^c\subset f^{-1}\left(0\right)$ and that f is continuous on $\mathbb{R}\setminus\{0\}$. Moreover f is $sem\mathcal{T}_{\Psi c}$-continuous at 0 as $E\subset f^{-1}\left(1\right)$.

4. c-density topology

We may follow here the proof of Theorem 2. Equivalently this can be deduced from inclusion $\mathcal{T}_c\supseteq\mathcal{T}_{a.e.}$ since $\mathcal{T}_{a.e.}=a.e.$-topology is completely regular and thus

$$\mathcal{T}_c\supseteq\mathcal{T}_{a.e.}=\mathcal{T}_{\text{sem}}=\mathcal{T}_{\text{ini}}.$$

5. $\mathcal{T}_{\mathcal{A}_I}$ topology.

For the convenience of the reader we recall the definition from the papers [Wo2] and [Wo3].

With $\mathcal{A}_I$ we denote the family of subsets of interval $[-1,1]$ having 0 as its I-density point.

Definition 5. (cf. [Wo2], [Wo3]) *We say that x is an $\mathcal{A}_I$-density point of $A \in S$, if for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_I$ such that the sequence*

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$$

of characteristic functions converges I-almost everywhere on B to 1.

We denote the set of all $\mathcal{A}_I$-density points $\Phi_{\mathcal{A}_I}(A)$ and the induced topology $\{A \in S : A \subset \Phi_{\mathcal{A}_I}(A)\}$ by $\mathcal{T}_{\mathcal{A}_I}$.

The argument here is analogous to that for $\mathcal{T}_I$ topology.

6. $\mathcal{T}_{\mathcal{A}_c}$-topology.

Let us recall now a definition from [Wo4]:

$\mathcal{A}_d-$ the family of measurable subsets of $[-1,1]$ that have Lebesgue density one at 0.

Definition 6. (cf. [Wo4]) *We say that x is an $\mathcal{A}_c$-density point of $A \in S$, if for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_d$ such that the sequence*

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(G(A)-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$$

of characteristic functions converges almost everywhere on B to 1.

We denote the set of all $\mathcal{A}_c$-density points by $\Phi_{\mathcal{A}_c}(A)$ and the induced topology $\{A \in S : A \subset \Phi_{\mathcal{A}_c}(A)\}$ by $\mathcal{T}_{\mathcal{A}_c}$.

The argument in this case is analogous to that for $\mathcal{T}_{\Psi c}$-topology. However, in the final part of the proof of the analogy of Theorem 2 we have to take one more subsequence. We shall show details below.

Let $x \in B$ where B is open in $sem\mathcal{T}_{\mathcal{A}_c}$ – a semi-regularization of $\mathcal{T}_{\mathcal{A}_c}$. By definition of semi-regularization there is a $\mathcal{T}_{\mathcal{A}_c}$-regular open set $A \subset B$ such that $x \in A$. Assume for simplicity that $x = 0$. We shall define a $sem\mathcal{T}_{\mathcal{A}_c}$-continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$ and $A^c \subset f^{-1}(0)$. By above $A = \Phi_{\mathcal{A}_c}(D)$ for some $D \in \mathcal{B}$. We have $\Phi_{\mathcal{A}_c}(D) = \Phi_{\mathcal{A}_c}(G(D))$, $G(D) \subset \Phi_{\mathcal{A}_c}(D) = A$ and $x \in \Phi_{\mathcal{A}_c}(D)$ i.e. for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_d$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges almost everywhere on B to 1. Hence the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ converges in measure to 1 on B. Since $G(D)$ is regular open in every interval $\left(\frac{1}{n+1}, \frac{1}{n}\right)$ we can find $\{I_k^n\}_{k=1}^{k_n}$ a finite number of open intervals,

$$I_k^n \subset G(D) \cap \left(\frac{1}{n+1}, \frac{1}{n}\right), \quad k = 1, \ldots, k_n$$

and closed intervals $\{J_k^n\}_{k=1}^{k_n}$, $J_k^n \subset I_k^n$ such that

$$\frac{\lambda\left(J_n' \cap \left(\frac{1}{n+1}, \frac{1}{n}\right)\right)}{\lambda\left(G\left(D\right)' \cap \left(\frac{1}{n+1}, \frac{1}{n}\right)\right)} < 1 + \frac{1}{n},$$

where $J_n = \bigcup\limits_{k=1}^{k_n} J_k^n$. The set $E = \bigcup\limits_{n=1}^{\infty} J_n$ forms a closed interval set and

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(G(E)-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$$

(as well as $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(E-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$) converges in measure to 1 on B. Consequently there is a subsequence

$$\left\{\chi_{\frac{1}{t_{n_{m_k}}}\cdot(G(E)-x)\cap[-1,1]}\right\}_{k\in\mathbb{N}}$$

that converges (as well as $\left\{\chi_{\frac{1}{t_{n_{m_k}}}\cdot(E-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$) almost everywhere on B to 1. Thus $x \in \Phi_{\mathcal{A}_c}\left(E\right)$.

We define an open interval set $V = \bigcup\limits_{n=1}^{\infty} I_n$, where $I_n = \bigcup\limits_{k=1}^{k_n} I_k^n$.

Define

$$f\left(x\right) = \begin{cases} 1 & x = 0, \\ \frac{dist(x,V^c)}{dist(x,V^c)+dist(x,E)} & x \neq 0. \end{cases}$$

It is easy to see that $f\left(0\right) = 1$, $B^c \subset V^c \subset f^{-1}\left(0\right)$ and that f is continuous on $\mathbb{R} \setminus \{0\}$. Moreover, f is $sem\mathcal{T}_{\mathcal{A}_c}$-continuous at 0 as $E \subset f^{-1}\left(1\right)$.

Problem 1. *Characterize these density topologies for which semi-regularization is the initial topology.*

References

[AM] R. A. Aleksandrian and E. A. Mirzahanian, *General Topology* (in Russian), Moscow 1979.

[AP] A. V. Arkhangel'skii and L. S. Pontryagin (Eds.) General Topology I, Encyclopaedia of Mathematical Sciences, Vol. 17, Springer-Verlag Berlin Heidelberg 1990.

[CLO] K. Ciesielski, L. Larson, and K. Ostaszewski, *I-Density Continuous Functions,* Memoirs of Amer. Math. Soc. **107**, no. 515 (1994).

[D] A. Denjoy, *Mémoire sur les dérivés des fonctions continues*, Journ. Math. Pures et Appl. **1** (1915), 105–240.

[GNN] C. Goffman, C.J. Neugebauer, and T. Nishura, *Density topology and approximate continuity*, Duke Math. J. **28** (1961), 497–505.

[GW] C. Goffman and D. Waterman, *Approximately continuous transformations*, Proc. Amer. Math. Soc. **12** (1961), 116–121.

[HP] O.Haupt and H. Pauc, *La topologie de Denjoy envisagée comme vraie topologie*, C. R. Acad. Sci. Paris **234** (1952), 390–392.

[M] H. Miller, *Baire outer kernels of sets*, Publications de l'Institut mathématique, Nouvelle série, **30** (44), (1981), 117–122.

[MR] M. Mrševič, I. L. Reilly, and M. K. Vamapurthy, *On semiregularization topologies*, J. Austral. Math. Soc. **38** (1985), 40–54.

[O] J. C. Oxtoby, *Measure and Category*, Graduate Texts in Math. 2 Springer-Verlag, New York, 1980.

[OM] R.J. O'Maley, *Approximately differentiable functions: the r topology*, Pacific J. Math. **72**, no. 1 (1977), 207–222.

[Os] K. Ostaszewski, *Continuity in the density topology*, Real Anal. Exchange **7**, no. 2 (1982), 259–270.

[PWW1] W. Poreda , E. Wagner-Bojakowska, and W. Wilczyński, *A category analogue of the density topology*, Fund. Math. **CXXV** (1985), 167–173.

[PWW2] W. Poreda, E. Wagner-Bojakowska, and W. Wilczyński, *Remarks on $\mathcal{I}$-density and $\mathcal{I}$-approximately continuous functions*, Comm. Math. Univ. Carolinae **26(3)**, (1985), 553–563.

[T] F. D. Tall, *The density topology*, Pacific J. Math. 62 (1) (1976), 275–284.

[W1] W. Wilczyński, *A generalization of density topology*, Real Anal. Exchange **8**, (1) (1982-83), 16–20.

[W2] W. Wilczyński, *Density Topologies*, in Handbook of Measure Theory, E. Pap, ed., Elsevier Science B. V. North Holland, Amsterdam 2002.

[WIWO] W. Wilczyński and W. Wojdowski, *A category ψ-density topology*, Cent. Eur. J. Math. **9** (2011), 1057–1066.

[Wo1] W. Wojdowski, *Density topology involving measure and category*, Demonstratio Mathematica, **22**, no. 3 (1989), 797–812.

[Wo2] W. Wojdowski, *A category analogue of the generalization of Lebesgue density topology*, Tatra Mountains. Publ. **42** (2009), 11–25.

[Wo3] W. Wojdowski, *Corrigendum in: A category analogue of the generalization of Lebesgue density topology*, Tatra Mountains Publ. (accepted).

[Wo4] W. Wojdowski, *A generalization of the c-density topology*, Tatra Mountains **62** (2015), 67–87.

[WT] M. Terepeta and E. Wagner-Bojakowska, *ψ-density topology*, Rendiconti Del Circolo Matematico di Palermo, Serie II, **XLVIII** (1999), 451–456.

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O SEMIREGULARYZACJI TOPOLOGII TYPU GĘSTOŚCIOWEGO

S t r e s z c z e n i e

Wykazujemy, że dla szeregu topologii gęstościowych semiregularyzacja topologii daje słabą topologię, dla której funkcje aporoksymatywnie ciągłe w tej topologii są ciągłe. Wskazujemy na różne metody dowodu tej własności i pytamy o charakteryzację topologii gęstości o tej własności.

Słowa kluczowe: punkt gęstości, topologia gęstości, semiregularyzacja

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Anna Makarewicz, Witold Mozgawa, and Piotr Pikuta

VOLUMES OF POLYHEDRA IN TERMS OF DETERMINANTS OF RECTANGULAR MATRICES

Summary

Formulas for volumes of octahedra, tetragonal pyramids, triangular prisms and truncated triangular prisms, involving determinants of rectangular matrices, are presented.

Keywords and phrases: determinant of rectangular matrix, volumes of polyhedra

1. Introduction

In [2] Radić introduced the following definition of the determinant of a rectangular matrix.

Definition 1.1. *Let $A = [A_1, A_2, \dots, A_n]$ be a $m \times n$ matrix with n columns $A_1, \dots, A_n$ and $m \leq n$. The determinant of A is defined as*

$$(1)\quad \det[A_1, A_2, \dots, A_n] = \sum_{1 \leq j_1 < \dots < j_m \leq n} (-1)^{r+j_1+j_2+\dots+j_m} \det[A_{j_1}, A_{j_2}, \dots, A_{j_m}],$$

where $r = 1 + 2 + \dots + m$.

Each column of a $m \times n$ matrix corresponds to a point in $\mathbb{R}^m$, so one can ask about the geometrical interpretation of the determinant 1 of a matrix consisting of n columns which contain coordinates of n points in $\mathbb{R}^m$. So far, very little is known about such interpretation, the only results include expressing areas of polygons in $\mathbb{R}^2$ in terms of determinants of $2 \times n$ matrices (see [3–6]), expressing volumes of "diamond-shaped" polyhedra in $\mathbb{R}^3$ in terms of determinants of $3 \times n$ matrices (see [6]), and the following theorem (see [6]).

Theorem 1.2. *Let $A_1, A_2, \ldots, A_{m+1}$ be vertices of an oriented m-simplex in $\mathbb{R}^m$. Then the m-dimensional volume of this simplex,* $\mathrm{vol}(A_1, A_2, \ldots, A_{m+1})$*, is equal to*

$$\mathrm{vol}(A_1 A_2 \ldots A_{m+1}) = \frac{1}{m!}\Big|\det[A_1, A_2, \ldots, A_{m+1}]\Big|.$$

In particular, for $n = 3$, we have the formula for the volume of a tetrahedron

$$\mathrm{vol}(A_1 A_2 A_3 A_4) = \frac{1}{6}\Big|\det[A_1, A_2, A_3, A_4]\Big|. \tag{2}$$

In this paper we concentrate on the geometrical interpretation of determinants of $3 \times n$ matrices and present formulas for volumes of octahedra, tetragonal pyramids, triangular prisms and truncated triangular prisms in terms of determinants of $3 \times n$ matrices.

2. Volumes of octahedra and tetragonal pyramids

Lemma 2.1. *Let $A_1, A_2, A_3, A_4, A_5, A_6$ be vertices of an octahedron* (see Fig. 1) *in $\mathbb{R}^3$ such that A_2, A_3, A_4, A_5 are consecutive vertices of a parallelogram. Then*

(a) *if the vertices A_1 and A_6 lie on opposite sides of the plane containing the parallelogram, then*

$$\det[A_1, A_2, A_3, A_4] \cdot \det[A_3, A_4, A_5, A_6] > 0,$$

(b) *if the vertices A_1 and A_6 lie on the same side of the plane containing the parallelogram, then*

$$\det[A_1, A_2, A_3, A_4] \cdot \det[A_3, A_4, A_5, A_6] < 0,$$

(c) $\det[A_1, A_2, A_3, A_4, A_5, A_6] = \det[A_1, A_2, A_3, A_4] + \det[A_3, A_4, A_5, A_6]$.

Proof. Let $B_1 = A_1 - A_2$, $B_3 = A_3 - A_2$, $B_4 = A_4 - A_2$ and $B_6 = A_6 - A_2$. Applying theorem 2, equation (14) from [6], and theorem 2.6 from [1], we have

$$\begin{aligned}\det[A_1, A_2, A_3, A_4] \cdot \det[A_3, A_4, A_5, A_6] &= \det[B_1, 0, B_3, B_4] \cdot \det[0, B_3, B_4, B_6] \\ &= \det[B_1, B_3, B_4] \cdot (-\det[B_3, B_4, B_6]) \\ &= -\langle B_3 \times B_4, B_1\rangle \cdot \langle B_3 \times B_4, B_6\rangle,\end{aligned}$$

where $\times$ and $\langle \cdot, \cdot \rangle$ denote the cross product and the scalar product of two vectors in $\mathbb{R}^3$, respectively. Since $B_3 \times B_4$ is a normal vector to the plane containing A_2, A_3, A_4, A_5, we have

$$\det[A_1, A_2, A_3, A_4] \cdot \det[A_3, A_4, A_5, A_6] > 0$$

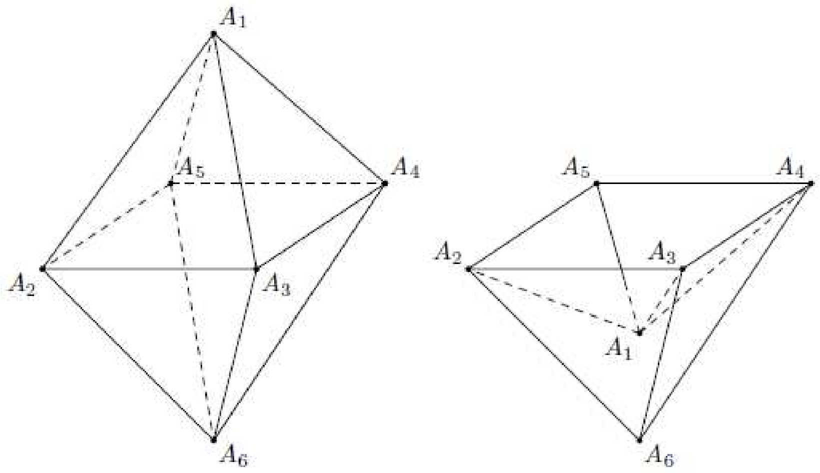

Fig. 1: Octahedra.

if and only if the vertices A_1 and A_6 lie on opposite sides of the plane containing A_2, A_3, A_4, A_5, and

$$\det[A_1, A_2, A_3, A_4] \cdot \det[A_3, A_4, A_5, A_6] < 0$$

if and only if the vertices A_1 and A_6 lie on the same side of the plane containing A_2, A_3, A_4, A_5. The proof of (a) and (b) is complete.

The proof of (c) requires the use of definition 1.1 and the property of a parallelogram

$$A_2 + A_4 = A_3 + A_5 \tag{3}$$

in the following calculations

$$\begin{aligned}
&\det[A_1, A_2, A_3, A_4, A_5, A_6] - \det[A_1, A_2, A_3, A_4] - \det[A_3, A_4, A_5, A_6]\\
&\quad = \det[A_1, A_2, A_5] - \det[A_1, A_2, A_6] - \det[A_1, A_3, A_5]\\
&\qquad + \det[A_1, A_3, A_6] + \det[A_1, A_4, A_5] - \det[A_1, A_4, A_6]\\
&\qquad + \det[A_1, A_5, A_6] + \det[A_2, A_3, A_5] - \det[A_2, A_3, A_6]\\
&\qquad - \det[A_2, A_4, A_5] + \det[A_2, A_4, A_6] - \det[A_2, A_5, A_6]\\
&\quad = \det[A_1, A_3 + A_5 - A_4, A_5] - \det[A_1, A_2, A_6] - \det[A_1, A_3, A_5]\\
&\qquad + \det[A_1, A_2 + A_4 - A_5, A_6] + \det[A_1, A_4, A_5] - \det[A_1, A_4, A_6]\\
&\qquad + \det[A_1, A_5, A_6] + \det[A_2, A_2 + A_4 - A_5, A_5] - \det[A_2, A_2 + A_4 - A_5, A_6]\\
&\qquad - \det[A_2, A_4, A_5] + \det[A_2, A_4, A_6] - \det[A_2, A_5, A_6]\\
&\quad = 0.
\end{aligned}$$

□

Theorem 2.2. *Let $A_1, A_2, A_3, A_4, A_5, A_6$ be vertices of a octahedron* (see Fig. 1) *in $\mathbb{R}^3$ such that A_2, A_3, A_4, A_5 are consecutive vertices of a parallelogram and the vertices A_1 and A_6 do not lie on the same side of the plane containing the parallelogram. Then the volume of the octahedron,* $\mathrm{vol}(A_1, A_2A_3A_4A_5, A_6)$*, is three times less than the absolute value of* $\det[A_1, A_2, A_3, A_4, A_5, A_6]$,

$$\begin{aligned}\mathrm{vol}(A_1, A_2A_3A_4A_5, A_6) &= \frac{1}{3}\Big|\det[A_1, A_2, A_3, A_4, A_5, A_6]\Big| \\ &= \frac{1}{3}\Big|\det[A_6, A_1, A_2, A_3, A_4, A_5]\Big|.\end{aligned}$$

Proof. It follows from lemma 2.1 and formula 2 that

$$\begin{aligned}\Big|\det[A_1, A_2, A_3, A_4, A_5, A_6]\Big| &= \Big|\det[A_1, A_2, A_3, A_4]\Big| + \Big|\det[A_3, A_4, A_5, A_6]\Big| \\ &= 3!\,\mathrm{vol}(A_1A_2A_3A_4) + 3!\,\mathrm{vol}(A_3A_4A_5A_6) \\ &= 3\,\mathrm{vol}(A_1, A_2A_3A_4A_5, A_6).\end{aligned}$$

Moreover, applying corollary 2.8 from [1] and formula 3, we have

$$\begin{aligned}&\det[A_1, A_2, A_3, A_4, A_5, A_6] \\ &\quad = -\det[A_6, A_1, A_2, A_3, A_4, A_5] + 2\det[A_1, A_2, A_3, A_4] - 2\det[A_1, A_2, A_3, A_5] \\ &\qquad +2\det[A_1, A_2, A_4, A_5] - 2\det[A_1, A_3, A_4, A_5] + 2\det[A_2, A_3, A_4, A_5] \\ &\quad = -\det[A_6, A_1, A_2, A_3, A_4, A_5],\end{aligned}$$

which completes the proof. □

Corollary 2.3. *Let A_1, A_2, A_3, A_4, A_5 be vertices of a tetragonal pyramid* (see Fig. 2) *in $\mathbb{R}^3$ such that A_2, A_3, A_4, A_5 are consecutive vertices of a parallelogram (which is the base of the pyramid), A_1 is the apex of the pyramid and B is an arbitrary point which lies on the plane containing the base of the pyramid. Then*

$$\begin{aligned}\mathrm{vol}(A_1, A_2A_3A_4A_5) &= \frac{1}{3}\Big|\det[A_1, A_2, A_3, A_4, A_5, B]\Big| \\ &= \frac{1}{3}\Big|\det[B, A_1, A_2, A_3, A_4, A_5]\Big|.\end{aligned}$$

Proof. Since the pyramid can be considered as a degenerate octahedron with an additional vertex B, the proof follows immediately from theorem 2.2. □

Corollary 2.4. *Let $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$ be vertices of a parallelepiped in $\mathbb{R}^3$ such that A_1, A_2, A_3, A_4 are consecutive vertices of a parallelogram and for some vector $T \in \mathbb{R}^3$ we have $A_{i+4} = A_i + T$, $i = 1, 2, 3, 4$. If B lies on the planes determined by A_1, A_2, A_3, A_4 and C lies on the planes determined by A_5, A_6, A_7, A_8 then the volume of this parallelepiped,* $\mathrm{vol}(A_1A_2A_3A_4, A_5A_6A_7A_8)$*, is equal to*

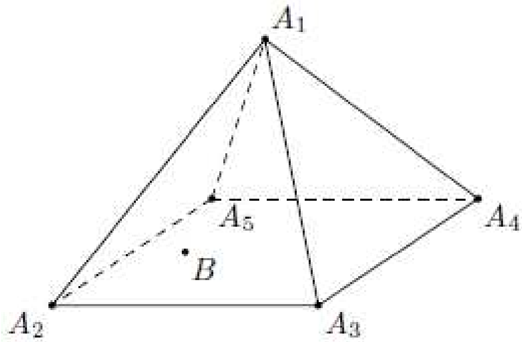

Fig. 2: A tetragonal pyramid.

$$\begin{aligned}\operatorname{vol}(A_1A_2A_3A_4, A_5A_6A_7A_8) &= \left|\det[B, A_1, A_2, A_3, A_4, A_{4+i}]\right| \\ &= \left|\det[A_i, A_5, A_6, A_7, A_8, C]\right|\end{aligned}$$

for every $i \in \{1, 2, 3, 4\}$.

Proof. The proof follows from the above corollary 2.3 and corollary 2.8 from [1]. □

Corollary 2.5. *Let* A_1, A_2, A_3, A_4, A_5 *be vertices of a pyramid* (see Fig. 2) *in* $\mathbb{R}^3$ *such that* A_1 *is the apex of the pyramid and* A_2, A_3, A_4, A_5 *are consecutive vertices of a parallelogram which is the base of the pyramid and lies on the plane* $\{(x, y, z)\colon z = 0\}$. *Then the volume of this pyramid,* $\operatorname{vol}(A_1, A_2A_3A_4A_5)$, *is three times less than the absolute value of* $\det[A_1, A_2, A_3, A_4, A_5]$,

$$\operatorname{vol}(A_1, A_2A_3A_4A_5) = \frac{1}{3}\left|\det[A_1, A_2, A_3, A_4, A_5]\right|.$$

Proof. It follows from corollary 2.3 and equation (14) from [6] that

$$\begin{aligned}\operatorname{vol}(A_1, A_2A_3A_4A_5) &= \frac{1}{3}\left|\det[A_1, A_2, A_3, A_4, A_5, O]\right| \\ &= \frac{1}{3}\left|\det[A_1, A_2, A_3, A_4, A_5]\right|.\end{aligned}$$

□

Lemma 2.6. *Let* $A_1, A_2, A_3, A_4, A_5, A_6$ *be vertices of an octahedron* (see Fig. 1) in $\mathbb{R}^3$ such that A_2, A_3, A_4, A_5 are consecutive vertices of a parallelogram and let $O = (0, 0, 0)$. Then

$$\det[A_1, A_2, A_3, A_4, A_5, A_6] = \det[A_1, A_2, A_3, A_4, A_5] - \det[A_2, A_3, A_4, A_5, A_6].$$

Proof. Applying definition 1.1 and equation 3, we have

$$\begin{aligned}
&\det[A_1, A_2, A_3, A_4, A_5, A_6] - (\det[A_1, A_2, A_3, A_4, A_5] - \det[A_2, A_3, A_4, A_5, A_6]) \\
&= -|A_1, A_2, A_6| + |A_1, A_3, A_6| - |A_1, A_4, A_6| + |A_1, A_5, A_6| \\
&\quad + |A_2, A_3, A_4| - |A_2, A_3, A_5| + |A_2, A_4, A_5| - |A_3, A_4, A_5| \\
&= |A_1, -A_2 + A_3 - A_4 + A_5, A_6| \\
&\quad + |A_3 - A_4 + A_5, A_3, A_4| - |A_2, A_3, A_5| + |A_2, A_3 + A_5 - A_2, A_5| - |A_3, A_4, A_5| \\
&= 0.
\end{aligned}$$

□

Corollary 2.7. *Let $A_1, A_2, A_3, A_4, A_5, A_6$ be vertices of an octahedron in $\mathbb{R}^3$ such that all the following conditions are satisfied:*

(a) A_2, A_3, A_4, A_5 *are consecutive vertices of a parallelogram,*

(b) $O = (0, 0, 0)$ *lies inside the octahedron,*

(c) A_1 *and* O *do not lie on the same side of the plane containing* A_2, A_3, A_4, A_5*,*

(d) A_6 *and* O *lie on the same side of the plane containing* A_2, A_3, A_4, A_5 (see Fig. 3).

Then

$$\begin{aligned}
\operatorname{vol}(A_1, A_2A_3A_4A_5, A_6) &= \frac{1}{3}\left(\Big|\det[A_1, A_2, A_3, A_4, A_5]\Big| + \Big|\det[A_2, A_3, A_4, A_5, A_6]\Big|\right) \\
&= \frac{1}{3}\Big|\det[A_1, A_2, A_3, A_4, A_5] - \det[A_2, A_3, A_4, A_5, A_6]\Big|
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{vol}(A_1, A_2A_3A_4A_5, O) &= \frac{1}{3}\Big|\det[A_1, A_2, A_3, A_4, A_5]\Big|, \\
\operatorname{vol}(O, A_2A_3A_4A_5, A_6) &= \frac{1}{3}\Big|\det[A_2, A_3, A_4, A_5, A_6]\Big|.
\end{aligned}$$

Proof. Applying theorem 2.2 and lemma 2.6, we have

$$\begin{aligned}
\operatorname{vol}(A_1, A_2A_3A_4A_5, A_6) &= \frac{1}{3}\Big|\det[A_1, A_2, A_3, A_4, A_5, A_6]\Big| \\
&= \frac{1}{3}\Big|\det[A_1, A_2, A_3, A_4, A_5] - \det[A_2, A_3, A_4, A_5, A_6]\Big|.
\end{aligned}$$

Decomposing an octahedron into two appropriate octahedra (see Fig. 3), we obtain

$$\begin{aligned}
&\operatorname{vol}(A_1, A_2A_3A_4A_5, A_6) = \operatorname{vol}(A_1, A_2A_3A_4A_5, O) + \operatorname{vol}(O, A_2A_3A_4A_5, A_6) \\
&= \frac{1}{3}\left(\Big|\det[A_1, A_2, A_3, A_4, A_5, O]\Big| + \Big|\det[O, A_2, A_3, A_4, A_5, A_6]\Big|\right) \\
&= \frac{1}{3}\left(\Big|\det[A_1, A_2, A_3, A_4, A_5]\Big| + \Big|\det[A_2, A_3, A_4, A_5, A_6]\Big|\right).
\end{aligned}$$

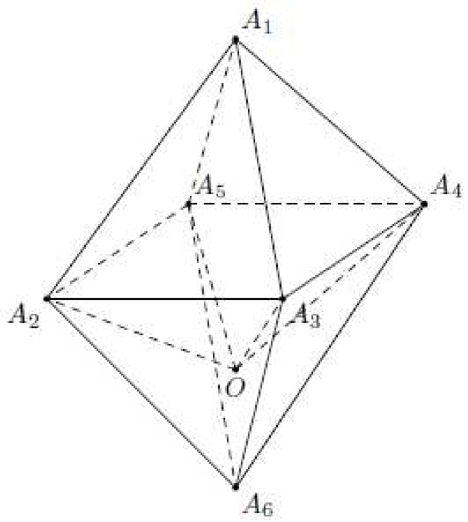

Fig. 3: An octahedron decomposed into two octahedra.

The other formulas follow from the above theorem 2.2 and equations (14)–(15) from [6]. □

3. Volumes of triangular prisms and truncated triangular prisms

Lemma 3.1. *Let $A_1, A_2, A_3, A_4, A_5, A_6$ be vertices of a triangular prism* (see Fig. 4) *in $\mathbb{R}^3$ such that $A_{i+3} = A_i + T$, $i = 1, 2, 3$, for some vector $T \in \mathbb{R}^3$. Then*

(a) $\det[A_1, A_2, A_3, A_4, A_5, A_6] = \det[A_1, A_2, A_3, A_4] + \det[A_3, A_4, A_5, A_6]$.

(b) $\det[A_1, A_2, A_3, A_4] = \det[A_3, A_4, A_5, A_6]$,

Proof. Applying definition 1.1, we obtain

$$\begin{aligned}
&\det[A_1, A_2, A_3, A_4, A_5, A_6] - \det[A_1, A_2, A_3, A_4] - \det[A_3, A_4, A_5, A_6] \\
&\quad = \det[A_1, A_2, A_2 + T] - \det[A_1, A_2, A_3 + T] - \det[A_1, A_3, A_2 + T] \\
&\qquad + \det[A_1, A_3, A_3 + T] + \det[A_1, A_1 + T, A_2 + T] - \det[A_1, A_1 + T, A_3 + T] \\
&\qquad + \det[A_1, A_2 + T, A_3 + T] + \det[A_2, A_3, A_2 + T] - \det[A_2, A_3, A_3 + T] \\
&\qquad - \det[A_2, A_1 + T, A_2 + T] + \det[A_2, A_1 + T, A_3 + T] - \det[A_2, A_2 + T, A_3 + T] \\
&\quad = 0.
\end{aligned}$$

Similarly, we have

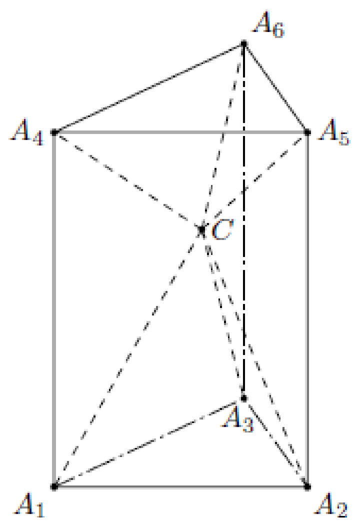

Fig. 4: A triangular prism.

$$
\begin{aligned}
&\det[A_1, A_2, A_3, A_4] - \det[A_3, A_4, A_5, A_6] \\
&\quad = \det[A_1, A_2, A_3] - \det[A_1, A_2, A_1 + T] + \det[A_1, A_3, A_1 + T] - \det[A_2, A_3, A_1 + T] \\
&\qquad - \Big(\det[A_3, A_1 + T, A_2 + T] - \det[A_3, A_1 + T, A_3 + T] + \det[A_3, A_2 + T, A_3 + T] \\
&\qquad\qquad - \det[A_1 + T, A_2 + T, A_3 + T] \Big) \\
&\quad = 0. \qquad \square
\end{aligned}
$$

Theorem 3.2. *Let $A_1, A_2, A_3, A_4, A_5, A_6$ be vertices of a triangular prism* (see Fig. 4) *in $\mathbb{R}^3$ such that $A_{i+3} = A_i + T$, $i = 1, 2, 3$, for some vector $T \in \mathbb{R}^3$. Then the volume of this prism,* $\mathrm{vol}(A_1A_2A_3, A_4A_5A_6)$*, is equal to*

$$
\begin{aligned}
\mathrm{vol}(A_1A_2A_3, A_4A_5A_6) &= \frac{1}{4}\Big| \det[A_1, A_2, A_3, A_4, A_5, A_6] \Big| \\
&= \frac{1}{2}\Big| \det[A_1, A_2, A_3, C] \Big| + \frac{1}{2}\Big| \det[A_4, A_5, A_6, C] \Big|, \\
&= \Big| \det[A_1, A_2, A_3, S] \Big|,
\end{aligned}
$$

where

$$
S = \frac{1}{6}\sum_{i=1}^{6} A_i
$$

is the center of mass of the prism and C is an arbitrary point which lies on either of the planes determined by A_1, A_2, A_3 and A_4, A_5, A_6, or between them.

Proof. The formulas

$$\begin{aligned}\mathrm{vol}(A_1A_2A_3, A_4A_5A_6) &= \frac{1}{2}\Big|\det[A_1,A_2,A_3,C]\Big| + \frac{1}{2}\Big|\det[A_4,A_5,A_6,C]\Big|,\\ &= \Big|\det[A_1,A_2,A_3,S]\Big|,\end{aligned}$$

follow from formula 2 for tetrahedron (see Fig. 4).

Letting $C = A_4$ and applying lemma 3.1, we have

$$\begin{aligned}\mathrm{vol}(A_1A_2A_3, A_4A_5A_6) &= \frac{1}{2}\Big|\det[A_1,A_2,A_3,C]\Big|\\ &= \frac{1}{4}\Big|\det[A_1,A_2,A_3,A_4] + \det[A_3,A_4,A_5,A_6]\Big|\\ &= \frac{1}{4}\Big|\det[A_1,A_2,A_3,A_4,A_5,A_6]\Big|.\end{aligned}$$

□

Lemma 3.3. *Let $A_1, A_2, A_3, A_4, A_5, A_6$ be vertices of a truncated triangular prism* (see Fig. 5) *in $\mathbb{R}^3$ such that the triangular faces are determined by A_1, A_2, A_3 and A_4, A_5, A_6, the vertices A_4, A_5, A_6 lie on the same side of the plane determined by A_1, A_2, A_3, and the edges joining A_i and A_{i+3} for $i = 1, 2, 3$ are parallel to each other. If*

$$S = \frac{1}{6}\sum_{i=1}^{6} A_i$$

is the center of mass of the prism then

$$\begin{aligned}\text{(a)}\ \det[A_1,A_2,A_3,S] &= -\frac{1}{2}\,(\det[A_1,A_2,A_4,S] + \det[A_2,A_3,A_5,S]\\ &\quad + \det[A_3,A_4,A_6,S])\\ &= \frac{1}{2}\,(\det[A_1,A_3,A_4,S] + \det[A_2,A_4,A_5,S]\\ &\quad + \det[A_3,A_5,A_6,S])\\ &= -\det[A_4,A_5,A_6,S],\end{aligned}$$

$$\begin{aligned}\text{(b)}\ \mathrm{sgn}(\det[A_1,A_2,A_4,S]) &= \mathrm{sgn}(\det[A_2,A_3,A_5,S]) = \mathrm{sgn}(\det[A_3,A_4,A_6,S])\\ &= -\,\mathrm{sgn}(\det[A_1,A_3,A_4,S]) = -\,\mathrm{sgn}(\det[A_2,A_4,A_5,S])\\ &= -\,\mathrm{sgn}(\det[A_3,A_5,A_6,S]),\end{aligned}$$

where sgn *denotes the signum function.*

Proof. Let

$$A_4 = A_1 + \alpha T, \quad A_5 = A_2 + \beta T, \quad A_6 = A_3 + \gamma T$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$ and $T \in \mathbb{R}^3$. We have

$$S = \frac{1}{3}(A_1 + A_2 + A_3) + \omega T,$$

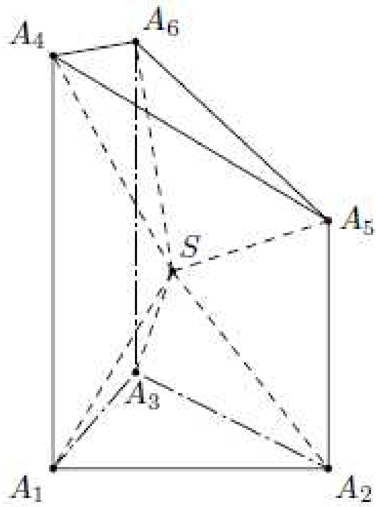

Fig. 5: A truncated triangular prism.

where we denoted

$$\omega = \frac{1}{6}(\alpha + \beta + \gamma).$$

Applying definition 1.1 we obtain

$$\begin{aligned}
\det[A_1, A_3, A_4, S] &= \det[A_1, A_3, A_1 + \alpha T] \\
&\quad - \det[A_1, A_3, \frac{1}{3}(A_1 + A_2 + A_3) + \omega T] \\
&\quad + \det[A_1, A_1 + \alpha T, \frac{1}{3}(A_1 + A_2 + A_3) + \omega T] \\
&\quad - \det[A_3, A_1 + \alpha T, \frac{1}{3}(A_1 + A_2 + A_3) + \omega T] \\
&= -\frac{1}{3}\left(\det[A_1, A_2, \alpha T] - \det[A_1, A_3, \alpha T] + \det[A_2, A_3, \alpha T]\right).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\det[A_1, A_2, A_4, S] &= \frac{1}{3}\left(\det[A_1, A_2, \alpha T] - \det[A_1, A_3, \alpha T] + \det[A_2, A_3, \alpha T]\right) \\
&= -\det[A_1, A_3, A_4, S],
\end{aligned}$$

and

$$\begin{aligned}
(4)\ \det[A_2, A_3, A_5, S] &= -\det[A_2, A_4, A_5, S] \\
&= \frac{1}{3}\left(\det[A_1, A_2, \beta T] - \det[A_1, A_3, \beta T] + \det[A_2, A_3, \beta T]\right), \\
(5)\ \det[A_3, A_4, A_6, S] &= -\det[A_3, A_5, A_6, S] \\
&= \frac{1}{3}\left(\det[A_1, A_2, \gamma T] - \det[A_1, A_3, \gamma T] + \det[A_2, A_3, \gamma T]\right).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}\det[A_1,A_2,A_3,S] &= \det[A_1,A_2,A_3]-\det[A_1,A_2,\frac{1}{3}(A_1+A_2+A_3)+\omega T]\\ &\quad+\det[A_1,A_3,\frac{1}{3}(A_1+A_2+A_3)+\omega T]\\ &\quad-\det[A_2,A_3,\frac{1}{3}(A_1+A_2+A_3)+\omega T]\\ &= -\det[A_1,A_2,\omega T]+\det[A_1,A_3,\omega T]-\det[A_2,A_3,\omega T]\\ &= -\frac{1}{2}(\det[A_1,A_2,A_4,S]+\det[A_2,A_3,A_5,S]+\det[A_3,A_4,A_6,S])\end{aligned}$$

and

$$\begin{aligned}\det[A_4,A_5,A_6,S] &= \det[A_1+\alpha T,A_2+\beta T,A_3+\gamma T]\\ &\quad-\det[A_1+\alpha T,A_2+\beta T,\frac{1}{3}(A_1+A_2+A_3)+\omega T]\\ &\quad+\det[A_1+\alpha T,A_3+\gamma T,\frac{1}{3}(A_1+A_2+A_3)+\omega T]\\ &\quad-\det[A_2+\beta T,A_3+\gamma T,\frac{1}{3}(A_1+A_2+A_3)+\omega T]\\ &= \det[A_1,A_2,\omega T]-\det[A_1,A_3,\omega T]+\det[A_2,A_3,\omega T]\\ &= -\frac{1}{2}(\det[A_1,A_3,A_4,S]+\det[A_2,A_4,A_5,S]+\det[A_3,A_5,A_6,S]).\end{aligned}$$

It follows from the above calculations that

$$\det[A_4,A_5,A_6,S]=-\det[A_1,A_2,A_3,S],$$

which completes the proof of (a).

To prove (b) we proceed analogously as in the proof of (a) and (b) of lemma 2.1. Let

$$A_1^{(4)}=A_1-A_4,\quad A_2^{(4)}=A_2-A_4,\quad A_5^{(4)}=A_5-A_4\quad\text{and}\quad S^{(4)}=S-A_4.$$

Applying theorem 2 from [6], theorem 2.6 from [1], and then equation (14) from [6], we have

$$\begin{aligned}\det[A_1,A_2,A_4,S]\cdot\det[A_2,A_4,A_5,S] &= \det[A_1^{(4)},A_2^{(4)},0,S^{(4)}]\cdot\det[A_2^{(4)},0,A_5^{(4)},S^{(4)}]\\ &= (-\det[A_2^{(4)},S^{(4)},A_1^{(4)}])\cdot(-\det[A_2^{(4)},S^{(4)},A_5^{(4)}])\\ &= \langle A_2^{(4)}\times S_4,A_1^{(4)}\rangle\cdot\langle A_2^{(4)}\times S_4,A_5^{(4)}\rangle.\end{aligned}$$

Since the center of mass S is situated inside the prism, A_1 and A_5 are located on opposite sides of the plane containing A_2, A_4 and S. Therefore

$$\operatorname{sgn}(\det[A_1,A_2,A_4,S])=-\operatorname{sgn}(\det[A_2,A_4,A_5,S]).$$

Using similar arguments, we conclude that

$$\begin{aligned}\operatorname{sgn}(\det[A_1,A_3,A_4,S]) &= -\operatorname{sgn}(\det[A_3,A_4,A_6,S]),\\ \operatorname{sgn}(\det[A_2,A_3,A_5,S]) &= -\operatorname{sgn}(\det[A_3,A_5,A_6,S]).\end{aligned}$$

To finish the proof of (b) we only need to use equations 4 and 5. $\square$

Theorem 3.4. *Let $A_1, A_2, A_3, A_4, A_5, A_6$ be vertices of a truncated triangular prism* (see Fig. 5) *in $\mathbb{R}^3$ such that the triangular faces are determined by A_1, A_2, A_3 and A_4, A_5, A_6, the vertices A_4, A_5, A_6 lie on the same side of the plane determined by A_1, A_2, A_3, and the edges joining A_i and A_{i+3} for $i = 1, 2, 3$ are parallel to each other. Then the volume of this prism,* $\mathrm{vol}(A_1A_2A_3, A_4A_5A_6)$*, is equal to*

$$\mathrm{vol}(A_1A_2A_3, A_4A_5A_6) = \Big|\det[A_1, A_2, A_3, S]\Big| = \Big|\det[A_4, A_5, A_6, S]\Big|,$$

where $S = \frac{1}{6}\sum_{i=1}^{6} A_i$ is the center of mass of the prism.

Proof. Decomposing the prism into eight tetrahedrons and then using formula 2 and lemma 3.3, we obtain

$$\begin{aligned}
6\,\mathrm{vol}(A_1A_2A_3, A_4A_5A_6) &= \Big|\det[A_1, A_2, A_3, S]\Big| + \Big|\det[A_4, A_5, A_6, S]\Big| \\
&\quad + \Big|\det[A_1, A_2, A_4, S]\Big| + \Big|\det[A_2, A_4, A_5, S]\Big| \\
&\quad + \Big|\det[A_1, A_3, A_4, S]\Big| + \Big|\det[A_3, A_4, A_6, S]\Big| \\
&\quad + \Big|\det[A_2, A_3, A_5, S]\Big| + \Big|\det[A_3, A_5, A_6, S]\Big| \\
&= 2\Big|\det[A_1, A_2, A_3, S]\Big| \\
&\quad + \Big|\det[A_1, A_2, A_4, S] + \det[A_2, A_3, A_5, S] + \det[A_3, A_4, A_6, S]\Big| \\
&\quad + \Big|\det[A_1, A_3, A_4, S] + \det[A_2, A_4, A_5, S] + \det[A_3, A_5, A_6, S]\Big| \\
&= 6\Big|\det[A_1, A_2, A_3, S]\Big|.
\end{aligned}$$

□

References

[1] A. Makarewicz, P. Pikuta, D. Szałkowski, *Properties of the determinant of a rectangular matrix*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **68**, no. 1 (2014), 31–41.

[2] M. Radić, *A definition of determinant of rectangular matrix*, Glas. Mat. **1**, no. 21 (1966), 17–22.

[3] M. Radić, *About a determinant of rectangular $2 \times n$ matrix and its geometric interpretation*, Beitr. Algebra Geom. **46**, no. 2 (2005), 321–349.

[4] M. Radić, *Areas of certain polygons in connection with determinants of rectangular matrices*, Beitr. Algebra Geom. **49**, no. 1 (2008), 71–96.

[5] M. Radić, *Certain equalities and inequalities concerning polygons in $\mathbb{R}^2$*, Beitr. Algebra Geom. **50**, no. 1 (2009), 235–248.

[6] M. Radić, R. Sušanj, *Geometrical meaning of one generalization of the determinant of a square matrix*, Glas. Mat. Ser. III **29**(49), no. 2 (1994), 217–233.

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OBJĘTOŚCI WIELOŚCIANÓW WYRAŻONE ZA POMOCĄ WYZNACZNIKÓW MACIERZY PROSTOKĄTNYCH

S t r e s z c z e n i e

W artykule podano, w jaki sposób można wyrazić objętości ośmiościanu, ostrosłupa czworokątnego, graniastosłupa trójkątnego i graniastosłupa trójkątnego ściętego za pomocą wyznaczników macierzy prostokątnych.

Słowa kluczowe: wyznacznik macierzy prostokątnej, objętość wielościanu

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to Professors J. Ławrynowicz and L. Wojtczak

Oliwia Chojnacka and Adam Lecko

ON THE DIFFERENTIAL SUBORDINATION
OF HARMONIC MEAN TO A LINEAR FUNCTION

Summary

In this paper we examine the differential subordination related to the harmonic mean. For a dominant being a linear function we improve the general result of [2].

Keywords and phrases: differential subordination, harmonic mean, arithmetic mean, geometric mean, convex function

1. Introduction

Let D be a domain in $\mathbb{C}$ and $\mathcal{H}(D)$ be the class of all analytic functions $f : D \to \mathbb{C}$. Let $\mathcal{H} := \mathcal{H}(\mathbb{D})$, where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ of functions f normalized by $f(0) := 0$ and $f'(0) := 1$, and $\mathcal{S}$ be the subclass of $\mathcal{A}$ of univalent functions.

A function $f \in \mathcal{H}$ is said to be subordinate to a function $F \in \mathcal{H}$ if there exists $\omega \in \mathcal{H}$ such that $\omega(0) := 0$, $\omega(\mathbb{D}) \subset \mathbb{D}$ and $f = F \circ \omega$ in $\mathbb{D}$. We write then $f \prec F$. When F is univalent, then

$$f \prec F \Leftrightarrow (f(0) = F(0) \wedge f(\mathbb{D}) \subset F(\mathbb{D}))\,. \tag{1.1}$$

Let $\beta \in [0, 1]$ and a, $b \in \mathbb{C}$. When $b + \beta(b - a) \neq 0$, the harmonic mean of a and b is given as

$$\frac{ab}{b + \beta(a - b)}.$$

Definition 1.1. *Let $\beta \in [0,1]$ and $\Phi \in \mathcal{H}(D)$. By $\mathcal{H}(\beta,\Phi)$ we denote the subclass of $\mathcal{H}$ of all nonconstant functions p such that $p(\mathbb{D}) \subset D$ and the function*

$$\mathbb{D} \ni z \mapsto \frac{p(z)(p(z)+zp'(z)\Phi(p(z)))}{p(z)+(1-\beta)zp'(z)\Phi(p(z))}$$

is either analytic or has only removable singularities with the analytic extension on $\mathbb{D}$.

For $\beta \in [0,1]$, $\Phi \in \mathcal{H}(D)$, $p \in \mathcal{H}(\beta,\Phi)$ and a univalent function $h \in \mathcal{H}$, in [3] and [2] it were started the studies of the differential subordination related to the harmonic mean of the type

$$\frac{p(z)(p(z)+zp'(z)\Phi(p(z)))}{p(z)+(1-\beta)zp'(z)\Phi(p(z))} \prec h(z), \quad z \in \mathbb{D}. \tag{1.2}$$

The above differential subordination with $\beta := 1/2$ and selected functions Φ and h was considered also in [6]. In this paper we continue the research of the differential subordination of the form (1.2) when h is a linear function.

Let us recall that the differential subordinations related to the arithmetic mean as well as to the geometric mean were studied by various authors. Given $\alpha \in [0,1]$, the differential subordination related to the arithmetic mean is given as follows:

$$p(z) + \alpha z p'(z)\Phi(p(z)) \prec h(z), \quad z \in \mathbb{D}.$$

The details and further references see [10, pp. 121–131]. The differential subordination related to the geometric mean was introduced in [5]. For further results in this direction see e.g., [7], [8], [9], [4] and [1]. We omit the details because the description requires some additional notation.

A function $f \in \mathcal{H}$ is said to be convex if it is univalent and $f(\mathbb{D})$ is a convex domain.

Let us we introduce the subclass $\mathcal{Q}$ (for details on corners of curves, see e.g., [11, pp. 51–65]). Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Definition 1.2. *By $\mathcal{Q}$ we denote the class of convex functions h with the following properties:*

(a) *$h(\mathbb{D})$ is bounded by finitely many smooth arcs which form corners at their end points (including corners at infinity),*

(b) *$E(h)$ is the set of all points $\zeta \in \mathbb{T}$ which corresponds to corners $h(\zeta)$ of $\partial h(\mathbb{D})$,*

(c) *$h'(\zeta) \neq 0$ exists at every $\zeta \in \mathbb{T} \setminus E(h)$.*

In [2] it was shown:

Theorem 1.3. *Let $\beta \in [0,1]$, $h \in \mathcal{Q}$ with $0 \in \overline{h(\mathbb{D})}$, and $\Phi \in \mathcal{H}(D)$ be such that $D \supset h(\mathbb{T} \setminus E(h))$ and*

$$\operatorname{Re}\Phi(h(\zeta)) \geq 0, \quad \Phi(h(\zeta)) \neq 0, \quad \zeta \in \mathbb{T} \setminus E(h). \tag{1.3}$$

If $p \in \mathcal{H}(\beta, \Phi)$, $p(0) = h(0)$ *and*

$$\frac{p(z)(p(z) + zp'(z)\Phi(p(z)))}{p(z) + (1-\beta)zp'(z)\Phi(p(z))} \prec h(z), \quad z \in \mathbb{D}, \tag{1.4}$$

then

$$p \prec h. \tag{1.5}$$

The assumption that $0 \in \overline{h(\mathbb{D})}$ as well as the condition (1.3) are essentially required for the proof of the above theorem. In what follows we improve the above theorem however with a linear function h and for selected functions Φ. In Theorem 2.2 with $\Phi \equiv 1$ we estimate the first coefficient $h'(0)$ of h for which $h(\mathbb{D})$ is a disk whose closure omits the origin however from (1.4) it follows (1.5). In Theorem 2.5 with $\Phi = 1/w,\ w \neq 0$, we do the same and moreover we find the range of $h'(0)$ where the assumption (1.3) is failed to be true, but (1.4) still implies (1.5).

2. Main results

Given $z_0 \in \mathbb{C}$ and $r > 0$, let $\mathbb{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ and $\mathbb{D}_r := \mathbb{D}(0, r)$. The lemma below is a special case of Lemma 2.2d [10, p. 22].

Lemma 2.1. *Let $h \in \mathcal{Q}$ and $p \in \mathcal{H}$ be a nonconstant function with $p(0) := h(0)$. If p is not subordinate to h, then there exist $z_0 \in \mathbb{D} \setminus \{0\}$ and $\zeta_0 \in \mathbb{T} \setminus E(h)$ such that*

$$p\left(\mathbb{D}_{|z_0|}\right) \subset h(\mathbb{D}), \tag{2.1}$$

$$p(z_0) = h(\zeta_0) \tag{2.2}$$

and for some $m \geq 1$,

$$z_0 p'(z_0) = m\zeta_0 h'(\zeta_0). \tag{2.3}$$

Write $\mathcal{H}_1(\beta)$ in case when $\Phi \equiv 1$. For $M \in (0, \beta/(2-\beta)]$ and h being a linear function the theorem below extends Theorem 1.3.

Theorem 2.2. *Let $\beta \in (0, 1]$. If $p \in \mathcal{H}_1(\beta)$, $p(0) := 1$,*

$$M \in (0, \beta/(2-\beta)] \cup [1, +\infty), \tag{2.4}$$

and

$$\left| \frac{p(z)(p(z) + zp'(z))}{p(z) + (1-\beta)zp'(z)} - 1 \right| < M, \quad z \in \mathbb{D}, \tag{2.5}$$

then

$$|p(z) - 1| < M, \quad z \in \mathbb{D}. \tag{2.6}$$

Proof. Fix $\beta \in (0,1]$ and M satisfying (2.4).

1. Let $h(z) := 1 + Mz,\ z \in \mathbb{D}$. Since h is univalent, $p(0) = h(0) = 1$ and (2.6) can be replaced by the inclusion $p(\mathbb{D}) \subset h(\mathbb{D}) = \mathbb{D}(1, M)$, by using (1.1) the condition (2.6) is equivalent to the subordination $p \prec h$.

When $M \geq 1$, then $0 \in \overline{h(\mathbb{D})} = \overline{\mathbb{D}(1, M)}$ and the assertion of the theorem follows directly from Theorem 1.3. However the proof which will demonstrate below confirms this fact again.

2. Suppose, on the contrary that p is not subordinate to h. Since $h \in \mathcal{Q}$ with $E(q) = \emptyset$, from Lemma 2.1 it follows that there exist $z_0 \in \mathbb{D} \setminus \{0\}$ and $\zeta_0 \in \mathbb{T}$ such that (2.1)–(2.3) hold. Thus

$$p(z_0) = 1 + M\zeta_0 \tag{2.7}$$

and for some $m \geq 1$,

$$z_0 p'(z_0) = mM\zeta_0. \tag{2.8}$$

Hence

$$\begin{aligned}
&\left| \frac{p(z_0)(p(z_0) + z_0 p'(z_0))}{p(z_0) + (1-\beta) z_0 p'(z_0)} - 1 \right| \qquad (2.9)\\
&= \left| \frac{p^2(z_0) + p(z_0) z_0 p'(z_0) - p(z_0) - (1-\beta) z_0 p'(z_0)}{p(z_0) + (1-\beta) z_0 p'(z_0)} \right| \\
&= \left| \frac{(1 + M\zeta_0)^2 + (1 + M\zeta_0) mM\zeta_0 - 1 - M\zeta_0 - (1-\beta) mM\zeta_0}{1 + M\zeta_0 + (1-\beta) mM\zeta_0} \right| \\
&= \left| \frac{M\zeta_0 + M^2\zeta_0^2 + mM^2\zeta_0^2 + \beta mM\zeta_0}{1 + (1 + (1-\beta)m) M\zeta_0} \right| \\
&= M \left| \frac{1 + \beta m + (1+m) M\zeta_0}{1 + (1 + (1-\beta)m) M\zeta_0} \right|.
\end{aligned}$$

Now we prove that

$$\left| \frac{1 + \beta m + (1+m) M\zeta_0}{1 + (1 + (1-\beta)m) M\zeta_0} \right| \geq 1, \tag{2.10}$$

which together with (2.9) contradicts (2.5). We have

$$\begin{aligned}
&|1 + \beta m + (1+m) M\zeta_0|^2 - |1 + (1 + (1-\beta)m) M\zeta_0|^2 \qquad (2.11)\\
&= (1+\beta m)^2 + 2M(1+m)(1+\beta m) \operatorname{Re}\left(\overline{\zeta_0}\right) + (1+m)^2 M^2 \\
&\quad - \left(1 + 2M(1 + (1-\beta)m) \operatorname{Re}\left(\overline{\zeta_0}\right) + (1 + (1-\beta)m)^2 M^2\right) \\
&= 2\beta m + \beta^2 m^2 + 2\left((1+m)(1+\beta m) - (1 + (1-\beta)m)\right) M \operatorname{Re}\left(\overline{\zeta_0}\right) \\
&\quad + \left((1+m)^2 - (1 + (1-\beta)m)^2\right) M^2 \\
&= \beta m \left((2 + \beta m) + 2(2+m) M \operatorname{Re}\left(\overline{\zeta_0}\right) + (2 + (2-\beta)m) M^2\right) \\
&\geq \beta m \left((2 + (2-\beta)m) M^2 - 2(2+m) M + (2 + \beta m)\right) \\
&= \beta m (2 + (2-\beta)m)(M-1) \left(M - \frac{2 + \beta m}{2 + (2-\beta)m}\right).
\end{aligned}$$

It remains to show that for every $m \geq 1$,

$$(M-1)\left(M - \frac{2+\beta m}{2+(2-\beta)m}\right) \geq 0. \tag{2.12}$$

Since the function

$$[1, +\infty) \ni m \mapsto \frac{2+\beta m}{2+(2-\beta)m}$$

is strictly decreasing and $(2+\beta)/(4-\beta) \leq 1$, for every $m \geq 1$ we have

$$\frac{\beta}{2-\beta} < \frac{2+\beta m}{2+(2-\beta)m} \leq \frac{2+\beta}{4-\beta} \leq 1.$$

Hence it follows at once that the inequality (2.12) holds. $\square$

When $\beta := 1/2$, we get the following result which in case of $M \in (0, 1/3)$ was shown in [6].

Corollary 2.3. *If* $p \in \mathcal{H}_1(1/2)$, $p(0) := 1$, $M \in (0, 1/3] \cup [1, +\infty)$ *and*

$$\left|\frac{2p(z)(p(z)+zp'(z))}{2p(z)+zp'(z)} - 1\right| < M, \quad z \in \mathbb{D},$$

then

$$|p(z) - 1| < M, \quad z \in \mathbb{D}.$$

Since $\mathcal{H}_1 \subset \mathcal{H}$, for $\beta := 1$ we have

Corollary 2.4. *If* $p \in \mathcal{H}$, $p(0) := 1$, $M > 0$ *and*

$$|p(z) + zp'(z) - 1| < M, \quad z \in \mathbb{D},$$

then

$$|p(z) - 1| < M, \quad z \in \mathbb{D}.$$

For $\beta \in (0, 1]$ and $\Phi(w) := 1/w,\ w \in \mathbb{C} \setminus \{0\}$, let $\mathcal{H}_2(\beta) := \mathcal{H}(\beta, \Phi)$. For $M \neq 1$ satisfying (2.13) and h being a linear function the theorem below extends Theorem 1.3.

Theorem 2.5. *Let* $\beta \in (0, 1]$. *If* $p \in \mathcal{H}_2(\beta)$, $p(0) := 1$,

$$M \in \left(0, \frac{\beta}{2-\beta}\right] \cup \left[1, \frac{1}{4}\left(6 - \beta + \sqrt{20(1-\beta)+\beta^2}\right)\right], \tag{2.13}$$

and

$$\left|\frac{p(z)\left(p(z) + \dfrac{zp'(z)}{p(z)}\right)}{p(z) + (1-\beta)\dfrac{zp'(z)}{p(z)}} - 1\right| < M, \quad z \in \mathbb{D}, \tag{2.14}$$

then

$$|p(z)-1|<M, \quad z\in\mathbb{D}.$$

Proof. Fix $\beta\in(0,1]$ and M satisfying (2.13). We repeat the argumentation of Part 1 of the proof of Theorem 2.2. Note that only for $M:=1$ we have $0\in\overline{h(\mathbb{D})}=\overline{\mathbb{D}(1.1)}$ and then the condition (1.3) is also true. For other M Theorem 1.3 can not be applied directly.

Suppose, on the contrary that p is not subordinate to h. Then the condition (2.1)–(2.3) and further (2.7)–(2.8) hold. Hence

$$\begin{aligned}(2.15)\qquad &\left|\frac{p(z_0)\left(p(z_0)+\dfrac{z_0p'(z_0)}{p(z_0)}\right)}{p(z_0)+(1-\beta)\dfrac{z_0p'(z_0)}{p(z_0)}}-1\right|\\ &=\left|\frac{p^3(z_0)+p(z_0)z_0p'(z_0))}{p^2(z_0)+(1-\beta)z_0p'(z_0)}-1\right|\\ &=\left|\frac{(1+M\zeta_0)^3+(1+M\zeta_0)mM\zeta_0-(1+M\zeta_0)^2-(1-\beta)mM\zeta_0}{(1+M\zeta_0)^2+(1-\beta)mM\zeta_0}\right|\\ &=\left|\frac{(1+M\zeta_0)^2M\zeta_0+mM^2\zeta_0^2+\beta mM\zeta_0}{(1+M\zeta_0)^2+mM\zeta_0-\beta mM\zeta_0}\right|\\ &=M\left|\frac{(1+M\zeta_0)^2+mM\zeta_0+\beta m}{(1+M\zeta_0)^2+mM\zeta_0-\beta mM\zeta_0}\right|.\end{aligned}$$

Now we prove that

$$(2.16)\qquad \left|\frac{(1+M\zeta_0)^2+mM\zeta_0+\beta m}{(1+M\zeta_0)^2+mM\zeta_0-\beta mM\zeta_0}\right|\geq 1.$$

which together with (2.15) contradicts (2.14). We have

$$\begin{aligned}(2.17)\qquad &\left|(1+M\zeta_0)^2+mM\zeta_0+\beta m\right|^2-\left|(1+M\zeta_0)^2+mM\zeta_0-\beta mM\zeta_0\right|^2\\ &=\left|(1+M\zeta_0)^2+mM\zeta_0\right|^2+\beta^2m^2\\ &\quad+2\beta m\,\mathrm{Re}\left\{(1+M\zeta_0)^2+mM\zeta_0\right\}\\ &\quad-\left|(1+M\zeta_0)^2+mM\zeta_0\right|^2-\beta^2m^2M^2\\ &\quad+2\beta m\,\mathrm{Re}\left\{\left((1+M\zeta_0)^2+mM\zeta_0\right)M\overline{\zeta_0}\right\}\\ &=\beta^2m^2-\beta^2m^2M^2\\ &\quad+2\beta m\,\mathrm{Re}\left\{\left(1+2M\zeta_0+M^2\zeta_0^2+mM\zeta_0\right)\left(1+M\overline{\zeta_0}\right)\right\}\\ &=\beta^2m^2-\beta^2m^2M^2+2\beta m\\ &\quad\times\mathrm{Re}\left\{1+M\overline{\zeta_0}+2M\zeta_0+2M^2+M^2\zeta_0^2+M^3\zeta_0+mM\zeta_0+mM^2\right\}\\ &=\beta^2m^2-\beta^2m^2M^2+2\beta m\left(1+2M^2+mM^2\right)\\ &\quad+2\beta mM\,\mathrm{Re}\left\{(M^2+3+m)\zeta_0+M\zeta_0^2\right\}\end{aligned}$$

$$= \beta m \big((4+2m-\beta m)M^2 + 2 + \beta m$$
$$+2M \operatorname{Re}\{(M^2+3+m)\zeta_0 + M\zeta_0^2\}\big).$$

Let $\zeta_0 := e^{i\theta_0},\ \theta_0 \in [0, 2\pi]$. Then

$$\operatorname{Re}\{(M^2+3+m)\zeta_0 + M\zeta_0^2\} = (M^2+3+m)\cos\theta_0 + M\cos(2\theta_0).$$

Consider the function

$$[0, 2\pi] \ni \theta \mapsto s(\theta) := (M^2+3+m)\cos\theta + M\cos(2\theta).$$

Note that $s'(\theta) = 0,\ \theta \in [0, 2\pi]$, if and only if

$$\big(M^2+3+m+4M\cos\theta\big)\sin\theta = 0,$$

i.e., if only if $\theta \in \{0, \pi, 2\pi\}$ or

$$\cos\theta = -\frac{M^2+3+m}{4M}. \tag{2.18}$$

But, since $m \geq 1$ and $M \neq 2$, which follows directly from (2.13), we have

$$\frac{M^2+3+m}{4M} \geq \frac{M^2+4}{4M} > 1.$$

Thus the equation (2.18) has no solution. Consequently,

$$\min_{\theta \in [0,2\pi]} s(\theta) = s(\pi) = -M^2 + M - 3 - m.$$

Hence and by (2.17) we obtain

$$\big|(1+M\zeta_0)^2 + mM\zeta_0 + \beta m\big|^2 - \big|(1+M\zeta_0)^2 + mM\zeta_0 - \beta m M\zeta_0\big|^2$$
$$\geq \beta m \big((4+2m-\beta m)M^2 + 2 + \beta m + 2M(-M^2+M-3-m)\big)$$
$$= \beta m \big(-2M^3 + (6+2m-\beta m)M^2 - (6+2m)M + 2 + \beta m\big)$$
$$= \beta m(M-1)\big(-2M^2 + (4+2m-\beta m)M - 2 - \beta m\big)$$
$$= -2\beta m(M-1)(M-M_1(m))(M-M_2(m)),$$

where

$$M_1(m) := \frac{1}{4}\left(4 + (2-\beta)m - \sqrt{16(1-\beta)m + (2-\beta)^2 m^2}\right)$$

and

$$M_2(m) := \frac{1}{4}\left(4 + (2-\beta)m + \sqrt{16(1-\beta)m + (2-\beta)^2 m^2}\right).$$

It remains to show that for every $m \geq 1$,

$$(M-1)(M-M_1(m))(M-M_2(m)) \leq 0. \tag{2.19}$$

As easy to check

$$0 < M_1(m) \leq 1, \quad M_2(m) > 1, \quad m \geq 1. \tag{2.20}$$

Moreover

$$M_1'(m) := \frac{1}{4}\left(2 - \beta - \frac{8(1-\beta) + (2-\beta)^2 m}{\sqrt{16(1-\beta)m + (2-\beta)^2 m^2}}\right), \quad m \geq 1.$$

Since the inequality

$$(2-\beta)\sqrt{16(1-\beta)m+(2-\beta)^2m^2} \le 8(1-\beta)+(2-\beta)^2m$$

equivalently written as

$$(2-\beta)^2\left(16(1-\beta)m+(2-\beta)^2m^2\right) \le \left(8(1-\beta)+(2-\beta)^2m\right)^2$$

is evidently true, so the function M_1 is decreasing with

$$\lim_{m\to+\infty} M_1(m)$$
$$= \lim_{m\to+\infty} \frac{4+2\beta m}{4+(2-\beta)m+\sqrt{16(1-\beta)m+(2-\beta)^2m^2}} = \frac{\beta}{2-\beta}.$$

Hence and by (2.20) we have

$$\frac{\beta}{2-\beta} < M_1(m) \le 1, \quad m \ge 1. \tag{2.21}$$

Since the function M_2 is obviously increasing so hence and by (2.20) for $m \ge 1$ we have

$$1 < \frac{1}{4}\left(6-\beta+\sqrt{20(1-\beta)+\beta^2}\right) = M_2(1) < M_2(m). \tag{2.22}$$

Taking now into account (2.21) and (2.22) we see that the inequality (2.19) holds which shows (2.16). □

For $\beta := 1$ we have

Theorem 2.6. *If* $p \in \mathcal{H}_2(1)$, $p(0) := 1$, $M \in (0, 3/2]$ *and*

$$\left|p(z)+\frac{zp'(z)}{p(z)}-1\right| < M, \quad z \in \mathbb{D},$$

then

$$|p(z)-1| < M, \quad z \in \mathbb{D}.$$

References

[1] N. E. Cho, O. Chojnacka, and A. Lecko, *On differential subordination of geometric mean*, Bull. Soc. Sci. Lettres Łódź **LXIV** (2014), no. 1, 11–24.

[2] N. E. Cho and A. Lecko, *Differential subordination of harmonic mean*, Comput. Methods Funct. Theory **15** (2015), no. 3, 427–437.

[3] O. Chojnacka and A. Lecko, *On some differential subordination of harmonic mean*, Bull. Soc. Sci. Lettres Łódź **LXIV** (2014), no. 2, 29–40.

[4] O. Crişan and S. Kanas, *Differential subordinations involving arithmetic and geometric means*, Appl. Math. Comp. **222** (2013), 123–131.

[5] S. Kanas, A. Lecko, and J. Stankiewicz, *Differential Subordinations and Geometric Means*, Complex Variables **28** (1996), 201–209.

[6] S. Kanas and A-E. Tudor, *Differential Subordinations and Harmonic Means*, Bull. Malay. Math. Sci. Soc. **38** (2015), 1243–1253.

[7] Y. C. Kim and A. Lecko, *On differential subordination related to convex functions*, J. Math. Anal. Appl. **235** (1999), 130–141.

[8] A. Lecko, *On differential subordinations and inclusion relation between classes of analytic functions*, Complex Variables **40** (2000), 371–385.

[9] A. Lecko and M. Lecko, *Differential subordinations of arithmetic and geometric means of some functionals related to a sector*, Int. J. Math. Math. Sci. **2011** (2011), no. Article ID 205845, 1–19.

[10] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker, Inc., New York, Basel, 2000.

[11] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin, Heidelberg, New York, 1992.

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O PODPORZĄDKOWANIU RÓŻNICZKOWYM ŚREDNIEJ HARMONICZNEJ FUNKCJI LINIOWEJ

S t r e s z c z e n i e

Dla $\beta \in [0,1]$ i a, $b \in \mathbb{C}$ takich, że $b+\beta(b-a) \neq 0$, wyrażenie $ab/(b+\beta(a-b))$ oznacza średnią harmoniczną liczb a i b. Dla $\beta \in (0,1]$ i $M > 0$ badane jest podporządkowanie różniczkowe

$$\frac{p(z)(p(z)+zp'(z)\Phi(p(z)))}{p(z)+(1-\beta)zp'(z)\Phi(p(z))} \prec 1+Mz, \quad z \in \mathbb{D},$$

w przypadku, gdy $\Phi \equiv 1$ oraz, gdy $\Phi(w) := 1//w,\ w \in \mathbb{C} \setminus \{0\}$. Wyznaczony jest zbiór tych M, dla których z powyższego podporządkowania wynika podporządkowanie

$$p(z) \prec 1+Mz, \quad z \in \mathbb{D}.$$

Słowa kluczowe: podporządkowanie różniczkowe, średnia harmoniczna, średnia arytmetyczna, średnia geometryczna, funkcja wypukła

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B U L L E T I N

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Contribution to the jubilee volume, dedicated to Professors J. Ławrynowicz and L. Wojtczak

Agnieszka Niemczynowicz and Agnieszka Bojarska-Sokołowska

INTEGRAL EQUATIONS IN PUZYNA'S TEACHING AND RESEARCH AS SEEN TODAY

Summary

In this paper, we discuss the important contribution of Józef Puzyna into development of the integral equations theory and its meaning today. We mention also the role of Puzyna for creating the famous Lwów Mathematical School as the occasion of celebration his 160 anniversary birthday.

Keywords and phrases: integral equations, Lwów mathematical School, Puzyna Józef

1. Introduction

The beginning of Polish Mathematical School come back to the XIX century. The higher school of Kraków and Lviv played a particular role in that period. Józef Puzyna – professor of Lviv University was the most meritorious person for that time. Let us mention the most important facts of his biography [1, 2, 4, 5].

Józef Puzyna was born in 1856 in Nowe Martynowo in Małopolska (today Ukraine). First he finished a grammar school in Lviv; after that he started his study at the University of Lviv (1875–1882), where passed his final teacher's exam in 1882. He took the doctor's degree of philosophy in 1883. Next, he continued his study at Berlin University (1883–1885). There were so famous scientists as K. Weierstrass, O. Fuchs, L. Kronecker and others, and Puzyna could attend to their lectures. These mathematicians exerted a powerful influence on the subject of his research work. In 1885 he was qualified as an assistant professor at University of Lviv, in 1889 he became associate professor and three years later full professor. In 1894/95 he was the dean of Department of Philosophy and during years 1904/05 he was the rector of University of Lviv. In 1900 he was chosen to be a correspondent member of Academy of Knowledge in Kraków. Since 1917 he was the chairman of Lviv Mathematical Society. His main interest was theory of integral equations and theory of analytic functions.

2. Puzyna's research work

The research work of Puzyna comes in disadvantage period for development of science in Poland. Puzyna worked in the Chair of Mathematics at University of Lviv. During a number years as a professor he fulfilled his obligations very conscientiously. His main effort was applied towards to deliver the best exhaustive selecton of lectures.

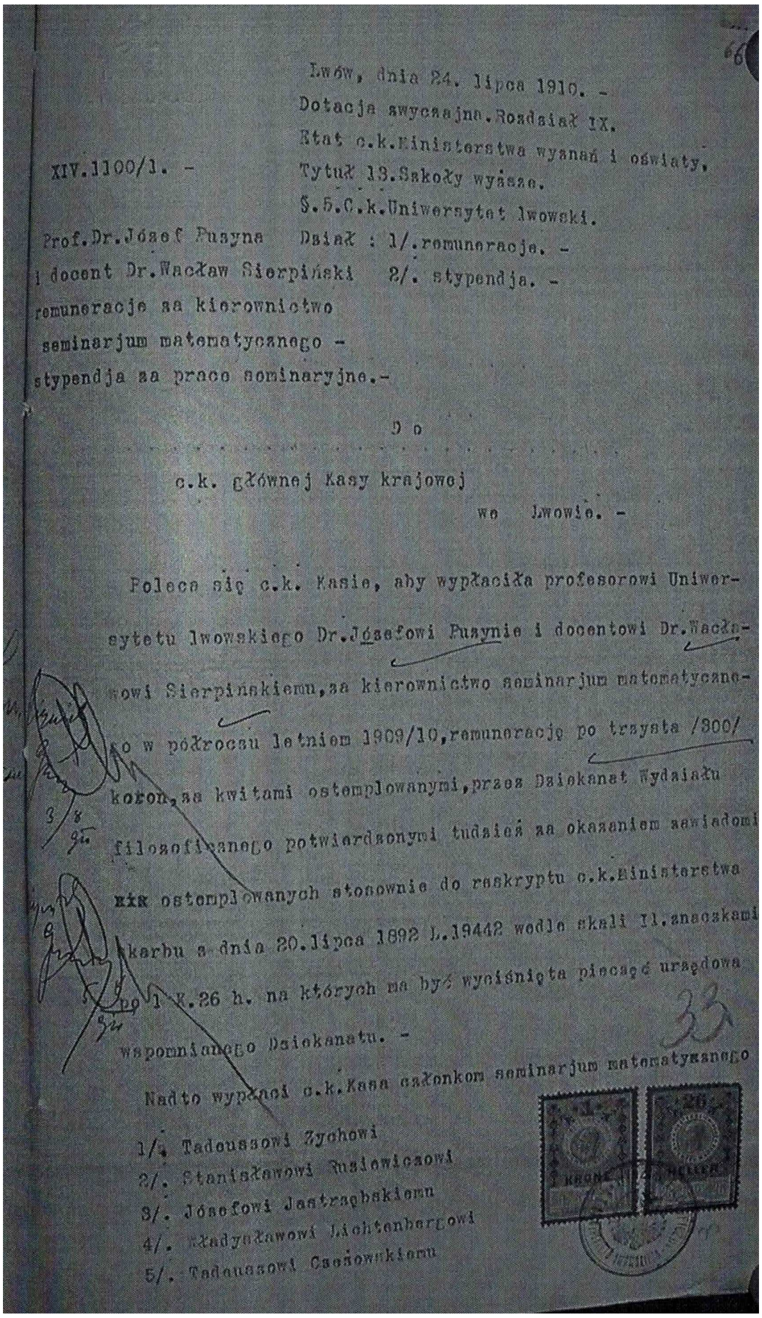
The proper research work of Puzyna developed stronger in the first period of his activity (till 1900). There were 15 research works and articles of that period of his life. They were published in Dissertations of Academy of Knowledge in Kraków, in Mathematical and Physical Works, in Monatshefte für Mathematik und Physik and the like at last main work of his life, two-volume monograph *Theory of analytic functions* (*Teorya funkcyj analitycznych*) published in 1898–1900. In that work he collected important research works of Weierstrass, Cauchy, Riemann and others, concerning analytic functions and information of set theory, topology, theory of groups and permutations as well [2, 3, 8–10]. Below we present the list of works of Puzyna [1]

1. *O pozornie dwuwartościowych określonych całkach podwójnych.* Pamiętniki Wydz. Matem.-Przyr. Akad. Umiej. vol. IX (1884) 1–15.
2. *O zastosowaniu uogólnionych form interpolacyjnych Lagrange'a.* ibid. vol. XIV (1888) 1–55.
3. *O tak zwanych miejscach skupienia i ich zastosowaniu w Analizie.* Muzeum IV (1888).
4. *Z Analizy* ibid.
5. *O pewnym twierdzeniu Foliego.* Pamiętniki Wydz. Matem.-Przyr. Akad. Umiej. vol. XVII (1889) 1–22.
6. *Prof. Wawrzyniec Żmurko; jego życie i dzieła.* Kosmos XIV.
7. *Kilka uwag o ogólnej teorji krzywych algebraicznych.* Rozpr. Akad. Umiej. vol. XXII (1891) 1–29.
8. *Über den Laguerre'schen Rang einer eindeutigen analytischen Function mit unendlich vielen Nullstellen.* Monatshefte für Mathematik und Physik vol. III (1892) 1–15.
9. *O wartościach funkcyj analitycznej na spółśrodkowych kręgach z kołem zbieżności jej elementu.* Rozpr. Akad. Umiej. vol. XXVI (1883) 200–204.

10. *Z teoryi n-krotnych całek określonych.* Prace matem.-fizyczne vol. IV (1892) 1–30.
11. *O rozwinięciach zbieżnych wewnątrz krzywych Cassini'ego.* ibid. vol. V (1894) 21–46.
12. *Über eine methodische Bildung der analytischen Ausdrücke $\sum f_v(x)$, $\sum f_v(x,y)$ von constanten Werten.* Monatshefte für Mathematik und Physik vol. V (1894) 67–84.
13. *O nierówności $g \geq |a_0|$.* Prace matem.-fizyczne vol. VI (1895) 1–4.
14. *Do teorji szeregów potęgowych.* Rozpr. Akad. Umiej. vol. XXXI (1896) 1–20.
15. *O twierdzeniu upraszczającym obliczanie czynników wykładniczych w Weierstrasowej teorji funkcyj eliptycznych.* Prace matem.-fizyczne vol. X (1898) 8–15.
16. *Teorja funkcyj analitycznych.* vol.I (1898), vol. II (1900) Lwów.
17. *O sumach nieskończenie wielu szeregów potęgowych i o twierdzeniu Mittag-Lefflera z teorji funkcyj.* Rozpr. Akad. Umiej. vol. XLIII (1903) 1–3.
18. *Geometrisches in der Weierstrassschen Theorie der algebraischen Funktionen.* Monatshefte für Mathematik und Physik vol. XX no. 1 (1909) 3–54, 193–241.
19. *Metoda wprowadzania całek logarytmicznych równania różniczkowego $\frac{dy}{dx} = \frac{ax+by+\dots}{a'x+by+\dots}$.* Księga pamiątkowa ku uczczeniu 250-tej rocznicy założenia Uniw. lwowskiego (1911) 3–8.
20. *O systemach krzywych z grupą preudoliniowych podstawień.* Rozpr. Akad. Umiej. vol. LI (19011) 1–124.
21. *Zastosowanie równań całkowych do tworzenia równań różniczkowych zwyczajnych rzędu 1-go i 2-go i różniczkowych cząstkowych rzędu 1-go.* Bulletin de l'Acad. des sciences de Cracovie (1913) 356–379.
22. *Zarys teorji równań całkowych.* Wektor vol. II no. 8-9, 356–370, 406–417.

3. Integral equations in Puzyna's didactical programme in the deductive and historical aspects

As a professor, Puzyna always encouraged his students for independent work and he supported them. Among students which worked under his protection were Franciszek Leja, Hugo Steinhaus, Antoni Łomnicki, Wacław Sierpiński, Stanisław Ruziewicz. An important role in development of the Lviv Mathematical School and Puzyna's teaching played the two kind of seminars (lower and higher) organized by Puzyna [4]. The partipants of these seminars were W. Lewicki, S. Ruziewicz, O. Nikodyn, A. Łomnicki. Puzyna persuaded W. Sierpinski to apply for habilitation at the University of Lviv, who was also a supervisor of seminar in 1910. Below on Fig. 1 we can see recommendation to pay gratification for J. Puzyna and W. Sierpiński for one of the seminars.


Lwów, dnia 24. lipca 1910. -
Dotacja zwyczajna. Rozdział IX.
Etat c.k. Ministerstwa wyznań i oświaty,
XIV.1100/1. - Tytuł 13. Szkoły wyższe.
§.5.C.k.Uniwersytet lwowski.
Prof.Dr.Józef Puzyna Dział : 1/.remuneracje. -
i docent Dr.Wacław Sierpiński 2/. stypendja. -
remuneracje za kierownictwo
seminarjum matematycznego -
stypendja za prace seminaryjne.-

Do
c.k. głównej Kasy krajowej
we Lwowie. -

Poleca się c.k. Kasie, aby wypłaciła profesorowi Uniwersytetu lwowskiego Dr.Józefowi Puzynie i docentowi Dr.Wacławowi Sierpińskiemu, za kierownictwo seminarjum matematycznego w półroczu letniem 1909/10, remuneracje po trzysta /300/ koron, za kwitami ostemplowanymi, przez Dziekanat Wydziału filozoficznego potwierdzonymi tudzież za okazaniem zawiadomień ostemplowanych stosownie do reskryptu c.k. Ministerstwa Skarbu z dnia 20. lipca 1892 L.19442 według skali II. znaczkami po 1 K.26 h. na których ma być wyciśnięta pieczęć urzędowa wspomnianego Dziekanatu. -

Nadto wypłaci c.k.Kasa członkom seminarjum matematycznego
1/. Tadeuszowi Zychowi
2/. Stanisławowi Ruziewiczowi
3/. Józefowi Jastrzębskiemu
4/. Władysławowi Lichtenbergowi
5/. Tadeuszowi Czeżowskiemu

Fig. 1: Recommendation to pay gratification for J. Puzyna and W. Sierpiński for one of the seminars.

Puzyna worked in the wide range of topics of lectures. There were lectures on extensive presentation of the introduction to the sections, of theory of series, complex numbers, set theory, substitutions, groups of invariants, integrals of Abel, elliptic functions, harmonic and modular functions. The following list represents the titles of lectures of Puzyna [1]

- Nowsze metody geometrji analitycznych [Modern Methods of Analytic Geometry] from 1885
- Nowa geometrja. I [Modern Geometry I] from 1885/86, II from 1886, III from 1901
- Teorja funkcyj analitycznych [Theory of Analytic Functions] from 1886/87
- Teorja funkcyj eliptycznych [Theory of Elliptic Functions] from 1887/88
- Teoryja funkcyj Abela [Theory of Abelian Functions] from 1888/89
- Rachunek różniczkowy i wstęp do Analizy [Differential Calculus and An Introduction to Analysis] from 1889/90
- Geometrja analityczna na płaszczyźnie [Analytic Geometry on Surface] from 1889/90
- Rachunek całkowy [Integral Calculus] from 1890/91
- Teorja liczb I [Theory of Numbers I] from 1891/92
- Teorja substytucyj [Theory of Substitutions] from 1892
- Rachunek przemienności [Calculus of commutativity] from 1893
- Geometrja różniczkowa [Differential Geometry] from 1902/03
- Niezmienniki [Invariants] from 1908/09
- Funkcje wielościanów, modułowe i eliptyczne [Functions of Polyhedrons, Modular and Elliptic] from 1908/09
- Całkowanie równań różniczkowych [Integration of Differential Equations] from 1910
- Odwzorowania cząsteczkowe [Conformal Mapping] from 1910
- Równania Fredholma [Fredholm's Equations] from 1908
- Wybrane ustępy z Algebry [Some Problems of Algebra]
- Geometja analityczna przestrzenna [Spatial Analytic Geometry]

- Równania różniczkowe Liego [Lie Differential Equations]
- Z historji Matematyki [On History of Mathematics]
- Gwiazdy Mittag-Lefflera [The Mittag-Leffler Stars]
- Geometrja nieeuklidesowa [Non-Euclidean Geometry]
- Równania różniczkowe cząstkowe [Partial Differential Equations]
- Krzywe algebraiczne [Algebraic Curves]
- Równania różniczkowe liniowe [Linear Differential Equations]

In the last period of his life, he concentrated his research work on theory of integral equations (a new mathematical area at that time). In 1907, during the X Conventions of Polish Naturalists and Physicians in Lviv, he delivered the report on *Relations between a continuous group of Lie and integral equations* (*O związku między grupami ciągłymi Liego a równaniami całkowymi*). In 1913 he published *Application of differential equations for constructing integral equations* (*Zastosowanie równań całkowych do tworzenia równań różniczkowych*) in Dissertations of Academy of Knowledge. He built differential equations, putting down for their integrals such conditions, that leading out of integral equations the solution. In the journal *Vector* he delivered an article under the title *Outline of the theory integral equations* (*Zarys teorji równań całkowych*). That article contained the lectures, published in 1913 as a supplementary course for teachers of secondary schools in Lviv. On the few pages he analyzed the important results about the integral equations based on works of such famous mathematicians as Fredholm, Hilbert, Schmidt, and others [11].

3.1. Kernel-Green function

Let us describe briefly the theory of integral equations presented by Puzyna in his works [11].

The equations of the form

$$f(s) = \varphi(s) - \lambda \int_a^b K(s,t)\varphi(t)dt, \tag{1}$$

$$\varphi(s) - \lambda \int_a^b K(s,t)\varphi(t)dt = 0 \tag{2}$$

are called *integral equations* or *Fredholm's equations,* where $K(s,t)$ is called *the kernel* of the integral equation and a and b are the limits of integration. We can easily observe that the unknown function $\varphi(t)$ appears under the integral sign. Type (1) represents *inhomogeneous* equations of the second order, type (2) – *homogeneous* equations of the first order. $K = K(s,t)$ supposed to be a finite continuous function of two independent real variables s,t in the segment $(s,t) = (a,b)$. The function $f(s)$ is continuous and finite in the segment $s = (a,b)$. The constant coefficients which are contained in $K(s,t)$ and in $f(s)$ are real. Parameter λ plays an important role in the theory of Fredholm's equations. It is unlimited and it can assume every real or imaginary value.

As we can see, the lecture starts from introduction to the simplest terminology of the theory of integral equations. At first he introduced the definitions of homogeneous and inhomogeneous integral equations (named by French mathematicians – Fredholm's equation). Further, he presented the methods of solving the integral equations containing the Kernel-Green function [11] in the case of degenerate kernel. At first, he considered the simplest form of the kernel.

He assumes for example, that in (1) he has

$$K(s,t) = A_1(s)B_1(t), \tag{3}$$

then the kernel is the product of two functions, the first one depends on s and the second depends on t. Then the equation (1) takes the form

$$f(s) = \varphi(s) - \lambda A_1(s) \int_a^b B_1(t)\varphi(t)dt. \tag{4}$$

Let

$$\int_a^b B_1(t)\varphi(t)dt = \text{const.} \tag{5}$$

Then

$$f(s) = \varphi(s) - \lambda A_1(s)C, \tag{6}$$

$$f(t) = \varphi(s) - \lambda A_1(t)C. \tag{7}$$

Substituting (7) into (3) he obtains

$$\int_a^b B_1(t)\left[f(t) + \lambda A_1(t)C\right]dt = C, \tag{8}$$

after putting

$$\begin{gathered} \int_a^b B_1 f(t)dt = (B_1, f) = (f, B_1) = \text{const.}, \\ \int_a^b B_1 A_1(t)dt = (B_1, A_1) = (A_1, B_1) = \text{const.} \end{gathered} \tag{9}$$

Assuming

$$(B_1, f) \neq 0, \qquad (B_1, A_1) \neq 0, \tag{10}$$

he obtains from (8)

$$C = \frac{(B_1, f)}{1 - \lambda(A_1, B_1)}. \tag{11}$$

Inserting calculated C in (6) he has

$$\varphi(s) = f(s) + \lambda A_1(s)\frac{(B_1, f)}{1 - \lambda(A_1, B_1)}. \tag{12}$$

When $\lambda = 1/(A_1, B_1)$, the equation does not have finite solution. Therefore, it is necessary reject the value of λ. Yet, when $\lambda = \frac{1}{(A_1, B_1)}$, it becomes a homogeneous equation

$$\varphi(s) = \lambda A_1(s) \int_a^b B_1(t)\varphi(t)dt \tag{13}$$

having a solution. Let λ be unrestricted in (13). Let introduce a constant C, then

$$\varphi(s) = \lambda A_1(s)C, \tag{14}$$

$$\varphi(t) = \lambda A_1(s)C. \tag{15}$$

Inserting (15) to (5) gives

$$\int_a^b B_1(t)[\lambda A_1(t)C]dt = C, \tag{16}$$

and hence $1 - \lambda(A_1, B_1) = 0$ or $\lambda = 1/(A_1, B_1)$ with that single value of λ the solution of equation exists (13) in the following form

$$\varphi(s) = \frac{A_1(s)}{A_1, B_1} \int_a^b B_1(t)\left[\frac{A_1(t)C}{(A_1, B_1)}\right] dt = \frac{CA_1(s)}{(A_1, B_1)^2}(A_1, B_1). \tag{17}$$

Takeing into account that C can be an unrestricted constant, the solution of (13) can take the form

$$\varphi(s) = C'A_1(s),$$

where C' is the unrestricted constant, except for value of λ, for which we do not have solution of the inhomogeneous equation, independent at all from that, how function $f(s)$ is, but depending only on the kernel and limit of integration.

3.2. Inhomogeneous equations with the general kernel

The next two steps, are to introduce the kernel in the general form and to find the relation between the given kernel and solving kernel.

Let

$$K(s,t) = A_1(s)B_1(t) + ... = A_n(s)B_n(t),\ n = 2, 3, ... \tag{18}$$

be an integral equation which we receive in the form

$$f(s) = \varphi(s) - \lambda \int_a^b \left(\sum_{\alpha=1}^{n} A_\alpha(s) B_\alpha(t) \right) \varphi(t) dt. \tag{19}$$

We put

$$\int_a^b B_\alpha(t)\varphi(t)dt = C_\alpha, \ \alpha = 1, 2, ... \tag{20}$$

From (19) we obtain

$$\begin{aligned} \varphi(s) &= f(s) + \lambda \sum_{\alpha=1}^{n} C_\alpha A_\alpha(s), \\ \varphi(t) &= f(t) + \lambda \sum_{\alpha=1}^{n} C_\alpha A_\alpha(t), \end{aligned} \tag{21}$$

where constants C_α needs to be determined, so that the equation (19) is satisfied. We obtain

$$\int_a^b B_\beta \left[f(t) + \lambda \sum_{\alpha=1}^{n} C_\alpha A_\alpha(t) \right] dt = C_\beta, \ \beta = 1, 2, ..., n. \tag{22}$$

Set

$$\begin{aligned} &\int_a^b B_\beta(t) f(t) dt = (B_\beta, f) = (f, B_\beta), \\ &\int_a^b B_\beta(t) A_\alpha(t) dt = (B_\beta, A_\alpha) = (A_\alpha, B_\beta) R_{\alpha,\beta} = K_{\alpha\beta}, \end{aligned} \tag{23}$$

for $\alpha, \beta = 1, 2, ..., n$. Then the relations (22) take the form

$$\begin{aligned} (1 - \lambda K_{11})C_1 - \lambda K_{12} C_2 - ... - \lambda K_{1n} C_n &= (B_1, f), \\ -\lambda K_{21} C_1 + (1 - \lambda K_{22}) C_2 - ... - \lambda K_{2n} C_n &= (B_2, f), \\ &\cdots \\ -\lambda K_{n1} C_1 - \lambda K_{n2} C_2 - ... + (1 - \lambda K_{nn}) C_n &= (B_n, f). \end{aligned} \tag{24}$$

In order to deduce finite and independent C_α form (24) it is necessary to satisfy the condition $D(\lambda) \neq 0$, where D denotes the determinant of (24). We assume, that λ differs from all the roots $\lambda_\alpha, \ \alpha = 1, 2, ..., \alpha$ of the equation $D(\lambda) = 0$, then C_α calculated from equation (24) takes the following form

$$C_\alpha = \left(\sum_{\beta=1}^{n} W_{\alpha,\beta(\lambda)}(B_\beta, f)\right) \cdot \frac{1}{D(\lambda)}, \quad \alpha = 1, 2, ..., n, \tag{25}$$

where $W_{\alpha,\beta}(\lambda)$ are the integral functions of values of parameter λ. Further we obtain

$$\begin{aligned}\varphi(s) = & f(s) + \frac{\lambda}{D(\lambda)} \left\{ \left[\sum_{\alpha=1}^{n} W_{\alpha_1}(\lambda) A_\alpha(s)\right](B_1, f) + ... \right. \\ & \left. + \left[\sum_{\alpha=1}^{n} W_{\alpha_n}(\lambda) A_\alpha(s)\right](B_n, f)\right\}.\end{aligned} \tag{26}$$

Yet $(B_\alpha, f) = \int B_\alpha(t)F(t)dt, \ \alpha = 1, 2, ...n$, and we can write

$$\varphi(s) = f(s) + \lambda \int_a^b K(s, t, \lambda) f(t) dt, \tag{27}$$

where

$$\begin{aligned}K(s, t, \lambda) = & \frac{1}{D(\lambda)} \left\{ \left[\sum_{\alpha=1}^{n} W_{\alpha_1}(\lambda) A_\alpha(s)\right](B_1, f) + ... \right. \\ & \left. + \left[\sum_{\alpha=1}^{n} W_{\alpha_n}(\lambda) A_\alpha(s)\right](B_n, f)\right\}.\end{aligned} \tag{28}$$

$K(s, t, \lambda)$ we call the kernel. It is independent of the function $f(s)$. Therefore, all inhomogenous equations with the kernel (18) and with the same limits of integration will have the same solving kernel.

3.3. Relation between the given kernel and solving kernel

We put into the given equation function $\varphi(s)$ represented in (27)

$$f(s) = \varphi(s) - \lambda \int_a^b K(s, t)\varphi(t) dt, \tag{29}$$

where $K(s, t)$ has the form (18). We obtain

$$\begin{aligned} f(s) = & f(s) + \lambda \int_a^b K(s, t, \lambda) f(t) dt \\ & -\lambda \int_a^b K(s, t) \left[f(t) + \lambda \int_a^b K(t, \tau, \lambda) f(\tau) d\tau \right] dt \end{aligned} \tag{30}$$

or

$$\int_a^b [K(s, t, \lambda) - K(s, t)] f(t) dt - \lambda \int_a^b K(s, t) \left[\int_a^b K(t, \tau, \lambda) f(\tau) d\tau \right] dt = 0. \tag{31}$$

We exchange the variables of integration t and τ in (31). Further, we obtain

$$\int_a^b \left[K(s,t,\lambda) - K(s,t) - \lambda \int_a^b K(s,t)K(t,\tau,\lambda)d\tau \right] dt = 0$$

for all $s = (a, ..., b)$. And consequently

$$K(s,t,\lambda) - K(s,t) - \lambda \int_a^b K(s,t)K(t,\tau,\lambda)d\tau = 0, \tag{32}$$

$$K(s,t,\lambda) - K(s,t) - \lambda \int_a^b K(t,\tau,\lambda)K(s,t)d\tau = 0. \tag{33}$$

As we can see, between the given kernel and the solving kernel there are always identical relations (32) and (33). Not repeated roots $\lambda = \lambda_\alpha$ of equation $D(\lambda) = 0$ are like that used in homogeneous equation giving finite solutions. On the other hand, the inhomogeneous equation becomes homogeneous when we assume that the function $f(s)$ is identical to zero. Then the equation (24) becomes

$$(B_\alpha, f) \equiv 0, \ \alpha = 1, 2, ..., n.$$

The equations become homogeneous and the condition that $C_\alpha \neq 0$ is $D(\lambda) = 0$.

Let λ' be a single root of the above equation. From (24) we get the solution in the form

$$C_1 = C_n D_1(\lambda'), \ C_2 = C_n D_2(\lambda'), \ldots, C_{n-1} = C_n D_{n-1}(\lambda'),$$

and return to (21); allowing $f(s) = 0$, we get

$$\varphi(s) = \lambda' C_n \sum_{\beta=1}^{n} D_\beta(\lambda')A_\beta(s), \ D_n(\lambda') = 1,$$

where $C_n = \text{const}$.

Let λ' be a double root of equation $D(\lambda) = 0$. Then in $D(\lambda') = 0$ for all subdeterminants – they vanish, and equations (24) give

$$C_1 = L_1(C_{n-1}, \lambda'), \ C_2 = L_2(C_{n-1}, C_n, \lambda'), \ldots, C_{n+2} = L_{n-2}(C_{n+1}, C_n, \lambda'),$$

where $L_1, \ldots, L_{n+2}$ are the linear homogeneous function of arbitrary C_{n+1}, C_n. Returning to (21), we get

$$\varphi(s) = C_{n+1}\psi(s,\lambda) + C_n\psi_2(s,\lambda). \tag{34}$$

In particular, for $C_{n-1} = 1, \ C_n = 0$ we have $\varphi(s) = \psi_1(s,\lambda')$ and for $C_{n-1} = 0, \ C_n = 1$ we have $\varphi(s) = \psi_2(s,\lambda')$.

Independently of the kernel $K(s,t)$, if we know the function $K(s,t,\lambda)$, which (with $K(s,t)$) satisfies the relations (32) or (33), then every inhomogeneous integral equation with the kernel $K(s,t)$ has the following solution

$$\varphi(s) = f(s) + \lambda \int_a^b K(s,t,\lambda)F(t)dt.$$

$K(s,t,\lambda)$ we call *solving kernel*. A kernel is here the so-called *Green's function.*

To sum up this part of considerations concerning theory of integral equations presented by Puzyna we can noticed that

- to a double root λ' correspond two particular solutions of the homogenous equation. The general solution is a homogeneous linear function (34) of these solutions (analogous conclusion, when roots repeat in equation $D(\lambda) = 0$ three, four times).
- All roots of the equations we call *basic values.*
- Every basic value λ' used in the homogenous equation give one finite solution of this equation. Every such solution we call *basic function* – it corresponds to this value λ'. Basic values and basic function depend only on kernel.

3.4. Fredholm's method

Now, follow Puzyna's example, we present *the Fredholms method* of solving integral equations.

Let us consider the determinant

$$\bar{D}_n(l) = \begin{vmatrix} 1 - lK_{11}, & -lK_{12}, & \dots, & -lK_{1n} \\ -lK_{21}, & 1 - lK_{22}, & \dots, & -lK_{2n} \\ & & \dots & \\ -lK_{n1}, & -lK_{n2}, & \dots, & 1 - lK_{nn} \end{vmatrix},$$

where $K_{\alpha\beta}$ are given finite quantities; l is an arbitrary parameter. For simplification, let

$$\begin{vmatrix} K\rho_1\rho_1, & K\rho_1\rho_2, & \dots, & K\rho_1\rho_r \\ K\rho_2\rho_1, & K\rho_2\rho_2, & \dots, & K\rho_2\rho_r \\ & & \dots & \\ K\rho_r\rho_1, & K\rho_r\rho_2, & \dots, & K\rho_r\rho_r \end{vmatrix} = K\begin{pmatrix} \rho_1, & \rho_2, & \dots, & \rho_r \\ \rho_1, & \rho_2, & \dots, & \rho_r \end{pmatrix}$$

be the determinant $D_n(l)$ developed according to the power of l; it takes the form

$$\bar{D}_n(l) = 1 - \sum_{\rho_1} K\begin{pmatrix} \rho_1 \\ \rho_1 \end{pmatrix} l + \sum_{\rho_1<\rho_2} K\begin{pmatrix} \rho_1 & \rho_2 \\ \rho_1 & \rho_2 \end{pmatrix} l^2 + \dots$$

$$+ (-1)^{n+1} \sum_{\rho_1<\rho_2<\dots<\rho_n} K\begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_n \\ \rho_1 & \rho_2 & \cdots & \rho_n \end{pmatrix} l^n.$$

In each of this sums we take ρ_1, ρ_2, …, ρ_n from the sequence of numbers 1, 2, …, n. Within the sum $\sum_{\rho_1<\rho_2<\dots<\rho_n}$ we specially notice one addend, namely

$$l^r K\begin{pmatrix} t_1, & t_2, & \dots, & t_r \\ t_1, & t_2, & \dots, & t_r \end{pmatrix}.$$

When in the determinant in question the elements have indices t_α, t_β from which the first indicates its number, whereas the second one – the number of the column, and

$(t_1, \ldots, t_n)$ has to be replaced by an unrestricted permutation, and the determinant does not change its value. These permutations are $r!$ When creating $r!$ determinants, in such a way, we obtain

$$l^r K\begin{pmatrix} t_1, & t_2, & \ldots, & t_r \\ t_1, & t_2, & \ldots, & t_r \end{pmatrix} = \frac{l^r}{r!}\sum_{r!} K\begin{pmatrix} t_1, & t_2, & \ldots, & t_r \\ t_1, & t_2, & \ldots, & t_r \end{pmatrix}. \tag{35}$$

Now we complete permutation into set all r^r variations with repetition. In these complementary comparisons it will be identical to t_α. The determinant which is created on the ground of these variations will be zero. Let us sum up all the vanishing determinants into the sum (35)

$$l^r K\begin{pmatrix} t_1, & t_2, & \ldots, & t_r \\ t_1, & t_2, & \ldots, & t_r \end{pmatrix} = \frac{l^r}{r!}\sum_{r^r} K\begin{pmatrix} t_1, & t_2, & \ldots, & t_r \\ t_1, & t_2, & \ldots, & t_r \end{pmatrix}.$$

Then we obtain

$$l^r \sum_{\rho_1<\rho_2<\ldots<\rho_n} K\begin{pmatrix} \rho_1, & \rho_2, & \ldots, & \rho_r \\ \rho_1, & \rho_2, & \ldots, & \rho_r \end{pmatrix} = \frac{l^r}{r!}\sum_{r^r} K\begin{pmatrix} \rho_1, & \rho_2, & \ldots, & \rho_r \\ \rho_1, & \rho_2, & \ldots, & \rho_r \end{pmatrix}, \tag{36}$$

where the sum on the right hand-side treats for all n^r variations with repetition created from numbers 1, 2, ..., n.

The function $K(s,t)$ is given as a function of two variables s, t independent, finite and continuous in the segment $(s,t) = (a, \ldots, b)$, $a < b$. We consider $t_{\rho_\alpha}, t_{\rho_\beta}$ within the Cartesian product of two segments, then $K(t_{\rho_\alpha}, t_{\rho_\beta})$ is value of function $K(s,t)$ at $(t_{\rho_\alpha}, t_{\rho_\beta})$. The number of these points amounts at n^2.

Let

$$K\begin{pmatrix} s_1, & s_2, & \ldots, & s_r \\ s_1, & s_2, & \ldots, & s_r \end{pmatrix} = \begin{vmatrix} K(s_1s_1), & K(s_1s_2), & \ldots, & K(s_1s_r) \\ & & \cdots & \\ K(s_rs_1), & K(s_rs_2), & \ldots, & K(s_rs_r) \end{vmatrix} \tag{37}$$

be function of r independent variables $s_1, s_2, \ldots, s_r \in (a, \ldots, b)$.

Let us define the points

$$t_1 = a,\ t_2 = a + \delta,\ \ldots,\ t_{n+1} = b,$$

where $\delta = t_{\rho_{k+1}} - t_{\rho_k} = \Delta t_{\rho_k}$, $k = 1, 2, \ldots, n$ and $l^r = \lambda^r \delta^r = \lambda^r \Delta t_{\rho_1} \ldots \Delta t_{\rho_n}$ for unrestricted parameter λ. Further we obtain

$$D_n(\lambda) = \begin{vmatrix} 1-\delta\lambda K(s_1s_1), & -\delta\lambda K(s_1s_2), & \ldots, & -\delta\lambda K(s_1s_n) \\ -\delta\lambda K(s_2s_1), & 1-\delta\lambda K(s_2s_2), & \ldots, & -\delta\lambda K(s_2s_n) \\ & & \cdots & \\ -\delta\lambda K(s_ns_1), & -\delta\lambda K(s_ns_2), & \ldots, & 1-\delta\lambda K(s_ns_n) \end{vmatrix}, \tag{38}$$

where $s_1 = a$, $s_2 = a + \delta, \ldots, s_n = b - \delta$. We can observe, that $D_n(\lambda)$ is absolutely eternally convergent series in parameter λ. Letting to the determinant $D_n(\lambda)$ to its limit, we can find the solution to the inhomogeneous equation of Fredholm

$$\varphi(s) = \frac{\lim\limits_{n\to\infty} D_{n,p}(\lambda)}{D(\lambda)},$$

when $D(\lambda) \neq 0$ and

$$D_{n,p}(\lambda) = \frac{\partial D_n(\lambda)}{\partial a_{1p}} f(s_1) + \ldots + \frac{\partial D_n(\lambda)}{\partial a_{np}} f(s_n),$$

where $a_{1p}, \ldots, a_{np}$ are the elements in the p-*th* row of the determinant $D_n(\lambda)$ and $s_1 = a + \delta, \ldots, s_n = a + n\delta$, with $\delta = \frac{b-a}{n}$.

4. Conclusions

The beginning of the theory of integral equations as an own discipline started at the late 19$^{\text{th}}$ and early 20$^{\text{th}}$ century work of Volterra, Fredholm and Hilbert. The lectures of Puzyna demonstrate the ideas of these pioneers. The Puzyna's goal was to present complete theory of the integral equations which was konown at that time, in transparent way. For this aim he mixed different methods to solve integral equations. The lecture is composed of such ideas:

1. Consider the equation (1) depending on the kernel, on the unknown function into the known function as a green's function.

2. Writing the determinant (37) of the resulting system (24).

3. Finding the solution inhomogeneous Fredholm's equation.

4. Follow by work of Fredholm [12], he showed, by used Hadamard's theorem, that
$$K\begin{pmatrix} s_1, & s_2, & \ldots, & s_r \\ s_1, & s_2, & \ldots, & s_r \end{pmatrix} \leq \sqrt{n^n} M^n.$$

Comparing the framework of Puzyna's lecture on integral equations with any modern lectures, we can notice, that his lecture was an introduction to the theory of integral equations mainly based on work of Fredholm. From the historical point of view the Fredholm's method was many years ahead of its time and, had one of the most famous follower who was D. Hilbert. His influence, we can notice in the Puzyna's lecture and in the concept of modern theory of integral equations, as well.

Nowadays the theory of integral equations is developing still and gives the opportunity to develop of other areas of mathematics. The preliminary course is constructed similar like Puzyna's lectures: they started from the same definitions, then went through the complete theory of Fredholm and Hilbert. More advanced theory usually appears as a part of course in functional analysis or as a help to write the boundary problems for ordinary and partial differential equations. An important role plays in numerical analysis of differential equations.

References

[1] A. Łomnicki, S. Ruziewicz, *Józef Puzyna,* Wiadom. Mat. 25 (1921), 113-119, (in Polish).

[2] S. Domaradzki, *The Growth of Mathematical Culture in the Lvov area in the Autonomy period (1870-1920)*, Matfyzpress, Prague 2011.

[3] A. Płoski, *O dziele Józefa Puzyny "Teorya funkcyj analitycznych"* [On the monograph of Józef Puzyna "Theory of Analytic Functions"], in Matematyka XIX wieku [Mathematics of the XIX century], ed. S. Fudali, Szczecin University, (1988), 237-244, (in Polish).

[4] Ya. G. Prytula, *Stefan Banach and Lviv Mathematical School,* in Functional Analysis and its Applications Proceedings of the International Conference on Functional Analysis and its Applications dedicated to the 110th Anniversary of Stefan Banach, May 28-31, 2002, Lviv, Ukraine, North-Holland Mathematics Studies, (2004), 1-3.

[5] Ya. G. Prytula, *Remarks on the history of mathematics in Lviv up to the middle of the XXth century,* Lvov Mathematical School in the Period 1915-45 as Seen Today, ed. by B. Bojarski, J. Ławrynowicz and Ya. G. Prytula, Banach Center Publications **87**, Institute of Mathematics, Polish Academy of Sciences, Warszawa, (2009), 17-26.

[6] R. Duda, *The Lvov School of Mathematics,* in The Global and Local: The History of Science and the Cultural Integration of Europe. Proceedings of the 2nd ICESHS, 6-9.09.2006 Kraków, ed. M. Kokowski.

[7] W. Sierpiński, *The Warsaw School of Mathematics and the Present State of Mathematics in Poland,* The Polish Review **4** no. 1-2, (1959), 51-63.

[8] S. Domoradzki, *Teoria mnogości w dziele Józefa Puzyny, Teorya funkcyj analitycznych,* Antiquitates Mathematicae **4** (2014), 45-58.

[9] S. Domoradzki, *O różnych aspektach działalności prof. J. Puzyny (1856–1919) we Lwowie,* [in:] J. Bečvař, M. Bečvařová (ed.), 35 Mezinárodni Konference Historie Matematiky, Matfyzpress, Univerzita Karlova, Praha 2014, 149-156.

[10] S. Domoradzki, M. Zarichnyi, *On some aspects of the set theory and topology in J. Puzyna's monumental work*, technical transactions, 1-NP/2014, 85-97.

[11] J. Puzyna, *Zarys teorji równań całkowych,* Wektor, vol. II, no.8-9, (1913), 356-370, 406-417.

[12] I. Fredholm, *Sur une classe d'equations fonctionnelles*, Acta Mathematica, **27(1)**, (1903), 365-390.

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RÓWNANIA CAŁKOWE W NAUCZANIU i PRACY BADAWCZEJ JÓZEFA KNIAZIA PUZYNY

S t r e s z c z e n i e

W artykule przedstawiono zarys wykładów Józefa Kniazia Puzyny dotyczących równań całkowych w kontekście jego nauczania oraz pracy badawczej.

Słowa kluczowe: równania całkowe, Lwowska Szkoła Matematyczna, Puzyna Józef kniaź

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